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Optimal $\mathcal{H}_\infty$ Design of Causal Multirate
Controllers and Filters

DISTRIBUTION

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2010
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Siblings,

and my wife Laura
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The airplane that brought me to the United States landed in Los Angeles airport (LAX) on September 15th, 2004. The first months in Irvine are very hard for any just arrived, and the fact that I barely spoke English at that time made them harder. Although none of my future projects involved going to graduate school, everything turned out to be different (for good) that initially planned. It’s been six years since I arrived and I am very happy and proud that these lines represent the acknowledgements of my Ph.D. dissertation at U.C. Irvine.

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Abstract of the Dissertation

Optimal \( \mathcal{H}_\infty \) Design of Causal Multirate Controllers and Filters

by

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Professor Athanasios Sideris, Chair

Multirate sampled data systems employ more than one sampling frequency. Three main representations of such systems are presented: lifted representation, polyphase representation and frequency response matrix of a multirate system. All of these are used in the design and analysis of multirate controllers.

A two-stage design methodology is presented for the design of \( \mathcal{H}_\infty \) optimal multirate controllers. The first stage comprises a fast rate \( \mathcal{H}_\infty \) optimal controller design and its decomposition. The second stage involves lifting and a second optimal design in slow rate. The design methodology is used two design examples to obtain either a SISO or a MIMO multirate \( \mathcal{H}_\infty \) optimal controller. The proposed approach lead to multirate controllers that well approximate the \( \mathcal{H}_\infty \) closed-loop performance of an optimal fast rate design at an overall computational load similar to multirate controllers designed by resampling the low frequency part of the fast rate controller.
An algorithm for the design of multirate causal $\mathcal{H}_\infty$ optimal controllers is presented. The controller design is based on the Youla parametrization of all stabilizing controllers. The causality of the controllers is assured by imposing constraints on the feedthrough term of its state space representation and the design problem is transformed into a constrained optimization problem, which is solved with the ellipsoid algorithm. The algorithm is validated through a controller design example.

The design of filter-banks is also studied. A lifting based design approach is proposed for the design of analysis and synthesis filter banks. The proposed methodology transforms the problem into a General Distance Problem where the filter bank designed is constrained to be causal. An iterative algorithm is also proposed to perform analysis and synthesis in the same design process. The results are validated with three filter-bank designs, which show small reconstruction error and same or better performance with lower filter complexity than some current design approaches.
Chapter 1

Introduction and Literature Review

Multirate sampled data systems and signals have been a subject of study for more than forty years. There is a wide variety of fields in which multirate systems are used. Chemical processes, image and speech compression, audio processing, Hard Disk Drive (HDD) control, and adaptive signal processing are some examples along many others. As a result of that research, many books have been published in the area of multirate digital systems and filter banks. Although one of the first books in the field [1] was published in 1983, two of the most cited in the current literature [2] and [3] belong to the early 90’s. Both of them present the fundamental elements in digital signal processing, multirate systems, filter banks and wavelets.

On the other hand, we have the \( \mathcal{H}_\infty \) control theory which develops the design of stabilizing controllers that achieve minimal closed loop \( \mathcal{H}_\infty \) norm [4], [5]. The parametrization of all the stabilizing controllers was first introduced by Youla et al. in [6] for Single Input Single Output (SISO) systems and extended to Multiple Input Multiple Output (MISO) systems in [7]. In addition, in [8] we can find a comprehensive treatment of the \( \mathcal{H}_\infty \) control theory and also complemented with
design examples. Such theory was initiated by Zames during the late 70’s and early 80’s. In [9] the problem of sensitivity reduction by feedback is formulated as a $\mathcal{H}_\infty$ norm optimization problem. Furthermore, the state-space solutions for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimization problems were derived in [10]. Other control techniques like Linear Quadratic Gaussian (LQG) control [11] or Linear Matrix Inequalities (LMI) [12], [13] have also been studied and developed to solve $\mathcal{H}_\infty$ control problems.

In $\mathcal{H}_\infty$ control theory, the design problem is formulated to meet certain frequency-domain specifications which also include robustness criteria [14]. There has been also research oriented to solve $\mathcal{H}_\infty$ control problems with time domain constraints [15] - [18] studying both SISO [19] and MIMO systems [20]. The optimal $\mathcal{H}_\infty$ controller design with time domain constrained is first transformed into a optimization problem over a set of convex constraints and later solved as a finite dimensional optimization problem. The numerical solution is found by using the Ellipsoid Algorithm [21], [22], which allows us to solve nondifferentiable optimization problems with a large number of variables and constraints.

With the development of the research in both multirate sampled data systems and $\mathcal{H}_\infty$ control theory, the interest in multirate control systems has also experienced a growth in the last four decades [23], [24]. Several representations of multirate systems have been proposed [25]. All of them take advantage of the periodicity of the multirate system. Although multirate sampled data systems employ more than one sampling frequency, they can be analyzed as a $N$-periodic Linear Periodic Time Varying (LPTV) system [26]. In addition, using the lifting technique in time domain we can derive a representation for multirate systems that uses Linear Time Invariant (LTI) systems [27]. A similar frequency domain approach obtains the so called in control theory frequency response matrix [28] which is also known as Alias Component (AC) matrix in digital signal processing [2].

The benefits of periodic compensation were discussed in [23], which shows that
for some robustness problems, periodic compensation is superior to the time invariant one. On the other hand, the issues on the multirate control were analyzed in [29]. Such issues include mainly oscillation between samples (known as intersample ripple), causality constraints in the controller and aliasing due to decimation. In [30] - [32] intersample ripple is eliminated by constraining the gains of the lifted controllers to be identical. The causality constraint was studied in [24], which is translated into the parametrization of all stabilizing controllers.

Since multirate controllers can have a component implemented at slow rate, they can reduce the computational load of the system to be implemented. An interesting control application studied in the literature is the one related to Hard Disk Drive (HDD) control. Much HDD related research has been developed by Horowitz et al. for the design and modeling of HDD’s ([33] and references therein). In addition, much work has also focused on the design of multirate controllers for HDD applications. Moreover, in [34] - [39] the computation savings of multirate controllers are analyzed and related aliasing content is measured. In [36] - [38] the slow rate part of the controller is implemented by using interlacing to make the computations uniform over the extent of the slow time period. More recent work [39] proposes a modified scheme that uses interlacing where the slow rate controller is designed so its output matches the fast rate controller at slow sample instances. This dissertation and related published work [40] proposes a two stage design approach for the obtention of $H_\infty$ optimal multirate controllers.

Finally, in digital signal processing, we find the design of multirate filter banks [2]. A digital multirate filter bank is formed by $M$ digital filters with either common input or output. The filter bank sharing the input is the so called analysis filter bank and it used to decompose a signal in subbands so it can be easily coded and transmitted. On the other hand, the filter bank with common output is known as the synthesis bank which is used after the transmission process to decode the
received signal and to reconstruct it by concatenating or adding as needed each subband. The idea of such system was first introduced in [41] and is also known as a $M$ channel filter bank. In order to improve the subband coding and reconstruction the synthesis and analysis filter are usually bandpass filters. If the reconstruction error is small, it is said that Perfect Reconstruction (PR) has been achieved by the filter banks [42]. Using the polyphase matrices [2] which are typically used in the digital signal processing, PR filter banks been also designed and analyzed in [42] and [43].

A different approach was introduced in [44] where the design of filter banks is formulated as an optimization problem. Using as a reference model a $m$ delay \((z^{-m})\), a model matching problem is solved using the $\mathcal{H}_{\infty}$ norm as the performance measure. Such delay is intended to model the elapsed time during the transmission of a signal between two points. Moreover, nonuniform multirate filter banks are designed in [45]. Nonuniform filter banks involved different sampling rates within the analysis and the synthesis filter bank. The approach used in such filter banks design is based on the polyphase matrices which establishes a control scheme that does not require the filters obtained to be causal. More precisely, it used the signal information from the past $m$ instances. However, that provokes a delay in the information used for the design which could lead to a loss in performance. The proposed work in this dissertation employs the lifting approach to overcome that possibility by using more updated signal information. The causal constraints that such approach require are also considered and solved in the design process using a constrained optimization algorithm.
1.1 Dissertation Outline

In the following lines we present the outline of this dissertation, which comprises five chapters and conclusions.

Chapter 2 establishes a theoretical basis for the rest of the chapters and it is mostly self contained. It presents the basic operators used in digital signal processing such as upsamplers, downsamplers, delays or lifting operators. Furthermore, using the presented operators to transform signals with different sampling periods to signals with common sampling rate, we obtain a representation for the multirate systems that uses Linear Time Invariant (LTI) systems. Such representations are obtained using the lifting technique. In addition, we also present an equivalent representation commonly used in digital signal processing, which is the polyphase representation. Equivalences between these two representations are derived and presented through a particular case. We also derive a representation for the multirate sampled data system that can be seen as a Linear Periodic Time Varying (LPTV) system, which is $N$-periodic. Similarly as the lifting technique in time domain, we obtain another representation of the multirate sampled data system which is known as the frequency response matrix of a multirate system. Such representation shows the transfer of signal content between different frequency bands in a LPTV system, which is also known as aliasing. We compute the distance from the multirate sampled data system to a fast rate system using as a measure the $\mathcal{H}_\infty$ norm of the difference between their respective frequency responses. Moreover, we also evaluate the aliasing content of a multirate digital system by computing the $\mathcal{H}_\infty$ norm of certain elements of its frequency response matrix.

Chapter 3 presents a two stage approach for the design of $\mathcal{H}_\infty$ multirate optimal controllers. Towards the reduction of the computational load involved in demanding applications such as hard disk drive (HDD) servoing, the objective is to design
a multirate $\mathcal{H}_\infty$ optimal controller that achieves performance as close as possible to that of the fast rate $\mathcal{H}_\infty$ optimal controller. Some work in recent years carry out the design in open loop without consideration of the closed loop specifications; there is one synthesis step in which the fast rate controller is designed and then it is modified (downsampled and interlaced) to a multirate system to reduce the computational load during real-time implementation. After such modification the multirate design is usually evaluated for excessive aliasing content due to downsampling. Although the use of notch filters could minimize aliasing in the design process and implementation, it would have to be tailored specifically for each system and application. Moreover, the addition of notch filters will affect optimality of the resulting design. In order to avoid these disadvantages, we propose a two-stage multirate $\mathcal{H}_\infty$ optimal controller design procedure. In the first stage, a fast rate optimal controller is designed by solving an appropriately formulated $\mathcal{H}_\infty$ optimal control problem. Next, the obtained fast rate controller is decomposed into its low and high frequency parts. Such decomposition can be done either using a series or a parallel configuration, which can be obtained for both SISO or MIMO systems. Then in the second stage, the high frequency part of the controller is retained as part of the problem formulation. The resulting $\mathcal{H}_\infty$ optimal control problem formulated is transformed to slow rate using the lifting technique presented in Chapter 2. Lifting transfers information in the design from fast rate to slow rate so that no unnecessary performance loss occurs due to downsampling. Most importantly, the slow rate redesign takes into account the possibility of aliasing in the system and can compensate for it. The second stage ends with the redesign of the low frequency part using the above mentioned lifted $\mathcal{H}_\infty$ optimal problem. We evaluate the design procedure performance using two HDD design examples, a SISO and a MIMO controller design. The performance evaluation is based on the frequency response matrix of the multirate system. It is used to compute the distance from the closed loop generated by the multirate controller to the one obtained using the fast rate
controller. It is also used to evaluate the aliasing content of the designed multirate closed loops. Furthermore, time domain simulations also show better performance in the multirate controllers that are obtained using the proposed design approach than the ones based by resampling. Finally, the performance loss and the computational load reduction are evaluated in terms of the multirate ratio $N$ used in the multirate $\mathcal{H}_\infty$ optimal controller design.

Chapter 4 provides a method that can be used for the design of causal multirate $\mathcal{H}_\infty$ optimal controllers. The design approach is based on the so called Youla parametrization which is used to obtain a parametrization of all stabilizing controllers for a given plant. We review also the causality constraints of multirate systems, which only affect the feedthrough term of the multirate system (i.e., $D$ matrix in the state-space representation). Generally speaking, such matrix has to be block lower triangular. For more general multirate systems the causal constraints redefines the block lower triangular condition. It has been proven [24] that the application of the causality condition in the solution generator (Youla parameter) in the Youla parametrization will lead to causal stabilizing controllers. The $\mathcal{H}_\infty$ optimal control problem can be then reformulated to an optimization problem with constraints, which can be solved using the Ellipsoid Algorithm. The proposed design algorithm starts by solving the unconstrained multirate $\mathcal{H}_\infty$ optimal control problem. The obtained optimal controller is tested to check if satisfies the causality condition which will fail in the general case. The following step in the design algorithm forces the designed controller to be causal by adding zeros where needed in its feedthrough term. Such causal controller is supplied as initial guess to the ellipsoid algorithm which is used now to solve the constrained optimization problem. The solution from the algorithm is then used to generate the causal multirate $\mathcal{H}_\infty$ optimal controller. The validation of the algorithm is done through a simple design of a SISO controller by varying the sampling period of the measurement and control signals. More specifi-
cally, we use the same system as in the SISO design example presented in Chapter 3. The performance of the causal multirate controllers is evaluated by computing the distance from their corresponding closed loop to the fast rate closed loop, which is based on the computation of their respective frequency response matrices. In addition, their aliasing content is also bounded and time domain simulations validate the results when comparing performance to that of the fast rate controller. Finally, we also compute the computation load reduction and the performance loss of the multirate $H_\infty$ optimal controllers with respect of the fast rate controller.

Chapter 5 presents the design of $M$ channel multirate filter banks and proposes a method for the design of the analysis (synthesis) filter bank based on the previous design of synthesis (analysis) filter bank. Based on these methods, an iterative design algorithm is also proposed so the multirate filter banks can be designed at once. The chapter also presents the general framework and previous work on the design of multirate filter banks, which are usually based on the concept of polyphase matrices in digital signal processing theory. We propose an algorithm based on the obtention of the lifted representation of a multirate system. The $H_\infty$ optimal control problem is formulated so the $H_\infty$ norm of the transfer function relating the input signal with the error between the desired signal and the reconstructed signal by the filter bank is minimized. Using the lifted representation, we modify the optimization problem to a General Distance Problem the solution of which provides a $H_\infty$ optimal, stable and causal lifted filter bank. The method proposed is validated through three examples, which are 2 channel multirate filter banks. The first two examples are extracted from the literature [44], being also typical design problems used as examples [2]. In these examples the analysis filters are given (predesigned) and the synthesis filters need to be designed by solving the $H_\infty$ optimal problem proposed above. The third example is used to validate the proposed iterative design algorithm; we start with randomly assigned analysis filters and with the objective
of designing the multirate filter bank (i.e., both analysis and synthesis filter bank). After several iterations a filter bank that locally minimizes the $\mathcal{H}_\infty$ norm of the error signal is obtained. The performance of the filter designed in the examples is evaluated by the achieved $\mathcal{H}_\infty$ norm of the error and also through a time domain simulation. In such simulation, the input signal used is a sinusoid with slow and fast frequency components. The error signal for the sinusoidal input is measured and its root mean square (rms) is also computed. Finally, the complexity of the designed filters is also analyzed. All the results show that the filters obtained are capable of reconstructing the desired signal with minimal error and with a relatively small computational complexity.

Finally, Chapter 6 discusses the conclusions and remarks of this dissertation and it also presents ideas for further research work along the lines of the research contributions presented.

## 1.2 Notation

We present in this section some of the notation that is used in this dissertation.

$A^T$ Transpose of $A$.

$\sigma(A)$ Largest singular value of $A$.

$\mathcal{H}_\infty$ The Hardy space of stable transfer functions $G(z)$.

$\mathcal{RH}_\infty$ Set of stable, proper, and real rational transfer functions $G(z)$ with state-space representation.

$\underline{G}(z)$ Lifted discrete time system obtained from $G(z)$.

$\underline{x}(k)$ Lifted discrete time signal obtained from $x(k)$. 
$G^\sim(z)$ Shorthand for $G^T(1/z)$.

$G_{\perp}(z)$ Orthogonal complement of the system $G(z)$.

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
Shorthand for the state-space representation of a discrete time system $G(z)$ thus $G(z) = C(zI - A)^{-1}B + D$. 
Chapter 2

Multirate systems

Sampled data systems which have different sampling rates in their inputs and outputs are known as Multirate Sampled Data (MRSD) systems [27]. Such multirate systems are presented as shift-varying operators in [26] which are also considered special cases of Linear Periodically Time Varying (LPTV) systems. Using the appropriate operators we can obtain alternative representations for which the multirate systems can be treated as $N$-periodic systems. On the other hand, LPTV systems have useful representations in terms of Linear Time Invariant (LTI) systems by using the lifting technique in time domain [27], linear switched time-varying systems [25], and alias-component or frequency response matrices [25]-[28].

If setup properly, basic operators used in multirate digital signal processing [3] as downsamplers, upsamplers, delays, and advances [2], can lead to the obtention of more complex ones as lifting or inverse lifting operators. All of these operators can also be used to compute practical representations of multirate systems. The representations based on LTI systems can be used in $H_\infty$ optimal control design problems and frequency response based representations can be used to measure the transfer between frequency bands (i.e., aliasing) in a multirate system since aliasing is an inherit property of multirate systems. Such property can be caused
by decimation since multirate systems employ more than one sampling rate.

The remainder of this Chapter is divided into three main parts. The first part (Section 2.1) briefly introduces the basic operations and tools used in digital signal processing. Such operators are used in the second part of the Chapter, which comprises Sections 2.2 through 2.4, to obtain three different representations of multirate systems. Section 2.2 is based on the lifting technique in time domain, which allows the representation of a multirate (i.e., LPTV) system with LTI systems. An equivalent representation is obtained in Section 2.3, which is derived from the polyphase components of the fast rate system. Moreover, a third representation is also obtained in Section 2.4. Such representation is known as the frequency response matrix of multirate systems and it is derived in frequency domain in a similar manner as the lifting technique was used in time domain to obtain the lifted representation in Section 2.2. The third part is contained in Section 2.5, which describes the basis of the analysis measures for multirate systems. Such measurements are the distance to a LTI system and the aliasing content [46] of a multirate system. Finally, Section 2.6 summarizes the results presented in this Chapter and provides an useful link with the following Chapters of this thesis.

2.1 Fundamentals on multirate systems

In this section we introduce the basic operations in multirate digital signal processing, which are the upsampler, downsampler, delay, advance, lifting and inverse lifting operators.
**Downsamplers and Upsamplers**

The most commonly used operators are the so-called downsamplers and upsamplers. Downsamplers take an input sequence $x(k)$ and produce the output sequence $y(k)$ as

$$y(k) = x(mk)$$  \hspace{1cm} (2.1)

where $m$ is an integer. Such operator is depicted in Figure 2.1a and an example of signal downsampling for $m = 2$ is shown in Figure 2.1b.

Similarly, upsamplers take an input sequence $x(k)$ and produce the output sequence $y(k)$ as

$$y(k) = \begin{cases} 
  x(k/n), & \text{if } k \text{ is an integer multiple of } n \\
  0, & \text{otherwise.}
\end{cases}$$  \hspace{1cm} (2.2)

where $n$ is an integer. The upsampler operator block is shown in Figure 2.1c and an example of signal upsampling for $n = 2$ is shown in Figure 2.1d.

![Figure 2.1: a) Downsampler $m$ block. b) Example for signal downsampling with $m = 2$. c) Upsampler $n$ block. d) Example for signal upsampling with $n = 2$.](image-url)
Delay and Advance

In addition to upsampling and downsampling, there are two operations commonly used in multirate signal processing; such operations delay or advance the content of signal on which they operate. Moreover, delay operators (i.e. $z^{-1}$) map a signal $x(k) = \{x(0), x(1), \ldots, x(k), \ldots\}$ to a signal $y_d(k) = x(k - 1)$ as follows

$$y_d(k) = \{0, x(0), \ldots, x(k - 1), \ldots\}$$

and advance operators (i.e. $z$) map $x(k) = \{x(0), x(1), \ldots, x(k), \ldots\}$ to signal $y_a(k) = x(k + 1)$ as follows

$$y_a(k) = \{x(1), x(2), \ldots, x(k + 1), \ldots\}.$$  

Lifting and Inverse Lifting

The lifting operator $L$ maps a sampled signal $\{x(0), x(1), \ldots, x(k), \ldots\}$ of period $T_f$ to a sampled signal $\{x(0), x(1), \ldots, x(k), \ldots\}$ of period $mT_f$. It holds that $\underline{x}(k) = [x^T(km) \ x^T(km + 1) \ldots x^T(km + m - 1)]^T$, where the lifted signal is underlined.

As shown in Figure 2.2b, we can obtain the lifting operator by combining both advance and downsample operators together. Mainly, the lifting operator is formed by an advance block chain and a cascade of downsamplers. Lifting operators transform discrete-time signals to higher dimension signals with higher period, that ensures that all the signal content is maintained through the operation.

Consider that each element of signal $x(k)$ is a column vector of dimension $n_x$
and that $m = 2$ thus we can express the vector lifting operation $\underline{x}(k) = Lx(k)$ as

\[
\begin{bmatrix}
  x(0) \\
  x(1) \\
  x(2) \\
  \vdots
\end{bmatrix}
= \begin{bmatrix}
  I_{n_x} & 0_{n_x} & 0_{n_x} & 0_{n_x} & 0_{n_x} & 0_{n_x} & \ldots \\
  0_{n_x} & I_{n_x} & 0_{n_x} & 0_{n_x} & 0_{n_x} & 0_{n_x} & \ldots \\
  0_{n_x} & 0_{n_x} & I_{n_x} & 0_{n_x} & 0_{n_x} & 0_{n_x} & \ldots \\
  0_{n_x} & 0_{n_x} & 0_{n_x} & I_{n_x} & 0_{n_x} & 0_{n_x} & \ldots \\
  0_{n_x} & 0_{n_x} & 0_{n_x} & 0_{n_x} & I_{n_x} & 0_{n_x} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
  x(0) \\
  x(1) \\
  x(2) \\
  x(3) \\
  x(4) \\
  x(5) \\
  \vdots
\end{bmatrix}
\]

Notice that the lifting operator $L$ is not causal and is time-varying. It can be proved (see for example [27]) that the lifting operator does not change the $l_2$ norm of the signal.

Similarly, the inverse lifting operator $(L^{-1})$ maps the lifted signal $y(k)$ of period $nT_f$ into its original form $y(k)$ of period $T_f$. As in the lifting operation, no signal content is lost in the inverse lifting process. We can obtain the inverse lifting operator
using a chain of delay blocks and a cascade of upsampler operators as shown in Figure 2.2d.

Consider that each element of signal \( y(k) \) is a column vector of dimension \( n_y \) and that \( n = 2 \) thus we can express the vector lifting operation \( \underline{y}(k) = L^{-1}y(k) \) as

\[
\begin{bmatrix}
y(0) \\
y(1) \\
y(2) \\
y(3) \\
y(4) \\
y(5) \\
\vdots
\end{bmatrix} =
\begin{bmatrix}
I_{n_y} & 0_{n_y} & 0_{n_y} & 0_{n_y} & 0_{n_y} & \cdots \\
0_{n_y} & I_{n_y} & 0_{n_y} & 0_{n_y} & 0_{n_y} & \cdots \\
0_{n_y} & 0_{n_y} & I_{n_y} & 0_{n_y} & 0_{n_y} & \cdots \\
0_{n_y} & 0_{n_y} & 0_{n_y} & I_{n_y} & 0_{n_y} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}
\begin{bmatrix}
y(0) \\
y(1) \\
y(2) \\
\vdots
\end{bmatrix}
\]

Notice that now the inverse lifting operator \( L^{-1} \) is causal and time-varying.

The operators presented in this section are used in the following to obtain the representation of multirate discrete-time systems.

### 2.2 Lifted representation of multirate systems.

Consider a continuous time system \( P(s) \) with \( q \) inputs \( u \) and \( p \) outputs \( y \) with dimensions \( d_u := \sum_{i=1}^{q} d_{u_i} \) and \( d_y := \sum_{j=1}^{p} d_{y_j} \) respectively. By holding and sampling its inputs and outputs respectively at different sampling rates, we can obtain a general multirate sampled data system. Such system is depicted in Figure 2.3.

Notice in Figure 2.3 that each of the zero-order hold (ZOH) and samplers have a different sampling time corresponding to the period of the signal they hold or sample. Since we are focused in the discrete multirate systems, in the following we obtain a digital multirate system by discretizing the continuous system \( P(s) \) and
modifying accordingly the zero-order hold and samplers.

Define the base (fast) sampling period $T_f$ as

$$T_f := \gcd(T_{n_1}, \ldots, T_{n_i}, \ldots, T_{n_q}, T_{m_1}, \ldots, T_{m_j}, \ldots, T_{m_p}) \quad (2.5)$$

where $\gcd$ means greatest common denominator and assume that

$$T_{n_i} := n_i T_f \quad \forall i = 1, \ldots, q.$$ 

and

$$T_{m_j} := m_j T_f \quad \forall j = 1, \ldots, p.$$ 

Next, we can obtain the system $P(z)$ which is discretized at the base rate $1/T_f$. More specifically, assuming that $P(s)$ has a minimal state-space representation as follows

$$P(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (2.6)$$

Therefore we can obtain the state-space representation of $P(z)$ as

$$P(z) = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} \quad (2.7)$$
where

\[ A_f = e^{T_f A} \quad (2.8) \]
\[ B_f = \int_0^{T_f} e^{\tau A} d\tau B \quad (2.9) \]
\[ C_f = C \quad (2.10) \]
\[ D_f = D \quad (2.11) \]

The general sampled data multirate system presented in Figure 2.3 becomes a multirate digital system by using the base rate \((1/T_f)\) discretized system \(P(z)\) and downsamplers and upsamplers as shown in Figure 2.4.

![Figure 2.4: Representation of a general multirate digital system.](image)

Notice from Figure 2.4 that the input signal \(u_i(k)\) has \(T_f\) sampling time and can be expressed as

\[ u_i(k) = \{u_i(0), u_i(T_f), \ldots, u_i(kT_f), \ldots\} \quad (2.12) \]

while the signal \(u_i(n_i k)\) has \(T_i = n_i T_f\) sampling time and therefore has the following form

\[ u_i(n_i k) = \{u_i(0), u_i(n_i T_f), \ldots, u_i(kn_i T_f), \ldots\}. \quad (2.13) \]

Such notation is equivalent for the output signals \(y_j(k)\) and \(y_j(m_j k)\) as shown in
(2.14) and (2.15).

\[ y_j(k) = \{y_j(0), y_j(T_f), \ldots, y_j(kT_f), \ldots\} \quad (2.14) \]
\[ y_j(m_j k) = \{y_j(0), y_j(m_j T_f), \ldots, y_j(km_j T_f), \ldots\}. \quad (2.15) \]

We remark that, in Figure 2.4, the upsampler and the zero order hold (ZOH) operators compose the \( u_i(k) \) signal by repeating \( n_i \) times each of the elements of the signal \( u_i(n_i k) \) while downsamplers keep the first of each \( m_j \) elements of \( y_j(k) \) to obtain \( y_j(m_j k) \).

As shown in Figure 2.5, using the lifting \((L_{\overline{m}_i})\) and inverse lifting \((L_{\overline{m}_i}^{-1})\) operators, the multirate system presented in Figure 2.4 can be transformed into a single rate \((NT_f)\) discrete system, where \( N \) is the multirate ratio of the multirate system. \( N \) can be computed as

\[ N = \text{lcm}(n_1, \ldots, n_i, \ldots, n_q, m_1, \ldots, m_j, \ldots, m_p) \quad (2.16) \]

where \( \text{lcm} \) stands for the least common multiple.

Define \( \overline{m}_i := \frac{N}{n_i} \forall i = 1, \ldots, q \) and \( \overline{m}_j := \frac{N}{m_j} \forall j = 1, \ldots, p \). Then the lifting operator \( L_{\overline{m}_j} \) maps the discrete time signal

\[ y_j(m_j k) = \{y_j(0), y_j(m_j T_f), \ldots, y_j(km_j T_f), \ldots\} \]

to a lifted signal

\[ \underline{y}_j(Nk) = \{\underline{y}_j(0), \underline{y}_j(NT_f), \ldots, \underline{y}_j(kNT_f), \ldots\} \]

of period \( NT_f \). It holds that

\[ \underline{y}_j(Nk) = [y_j^T(kNT_f), y_j^T((kN + m_j)T_f), \ldots, y_j^T((kN + (\overline{m}_j - 1)m_j)T_f)]^T \]

Similarly, the inverse lifting operator \( L_{\overline{m}_i}^{-1} \) maps the lifted signal \( \underline{u}_i(Nk) \) into the signal \( u_i(n_i k) \).
Consider the discrete time system $P(z)$ computed in (2.9)-(2.11) with minimal state-space realization as follows

$$P(z) = \begin{bmatrix} A_f & B_{1f} & B_{2f} & \cdots & B_{if} & \cdots & B_{qf} \\ C_{1f} & D_{11f} & D_{12f} & \cdots & D_{i1f} & \cdots & D_{q1f} \\ C_{2f} & D_{21f} & D_{22f} & \cdots & D_{i2f} & \cdots & D_{q2f} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ C_{jf} & D_{j1f} & D_{j2f} & \cdots & D_{jif} & \cdots & D_{jqf} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ C_{pf} & D_{p1f} & D_{p2f} & \cdots & D_{pif} & \cdots & D_{pqf} \end{bmatrix}$$

(2.17)

Moreover we consider the state-space realization of the system $P_{ji}(z)$ which correspond to the transfer function from the $i$th input to the $j$th output as

$$P_{ji}(z) = \begin{bmatrix} A_f & B_{if} \\ C_{jf} & D_{jif} \end{bmatrix}$$

(2.18)

and consequently we can represent the lifted system $P_{ji}(z)$ as shown in Figure 2.6.

In order to obtain the state-space representation of the lifted system $P_{ji}(z)$, we follow a similar approach as in [27] and [26]. Consider the base rate $(1/T_f)$ discretized system $P_{ji}(z)$ which state-space realization is expressed in (2.18) and the scheme presented in Figure 2.7.
The operators $H_{n_i}$ and $S_{m_j}$ map the signals $u_{n_i}(Nk)$ to $\tilde{u}_{n_i}(Nk)$ and $\tilde{y}_{m_j}(Nk)$ to $\tilde{y}_{m_j}(Nk)$ respectively and are defined as follows:

$$H_{n_i} := \text{block diag} \left\{ \begin{bmatrix} I \\ \vdots \\ I \end{bmatrix}_{n_id_{u_i} \times d_{u_i}} \right\}$$

$$S_{m_j} := \text{block diag} \left\{ \begin{bmatrix} I & 0 & \ldots & 0 \\ \vdots \\ I \end{bmatrix}_{d_{yj} \times m_j d_{yj}} \right\}$$

Note that the blocks are repeated diagonally $\overline{n}_i$ and $\overline{m}_j$ times for $H_{n_i}$ and $S_{m_j}$ respectively. Therefore the operators $H_{n_i}$ and $S_{m_j}$ have dimensions $N d_{u_i} \times \overline{n}_i d_{u_i}$ and $\overline{m}_j d_{yj} \times N d_{yj}$ respectively.

We remark here that the input signals in Figure 2.7 are defined as

$$\tilde{u}(Nk) = [u_i^T(kNT_f), u_i^T((kN + 1)T_f), \ldots, u_i^T((kN + N - 1)T_f)]^T$$

$$u_i(Nk) = [u_i^T(kNT_f), u_j^T((kN + n_i)T_f), \ldots, u_i^T((kN + (\overline{n}_i - 1)n_i)T_f)]^T$$
and the output signals follow a similar notation, thus
\[
\tilde{y}_j(Nk) = \left[ y_j^T(kNT_f), y_j^T((kN + 1)T_f), \ldots, y_j^T((kN + N - 1)T_f) \right]^T
\] (2.23)
\[
y_j(Nk) = \left[ y_j^T(kNT_f), y_j^T((kN + m_j)T_f), \ldots, y_j^T((kN + (m_j - 1)m_j)T_f) \right]^T
\] (2.24)

Consider that the lifted system \( \tilde{P}_{ji}(z) \) has a minimal state-space representation
\[
\tilde{P}_{ji}(z) = \begin{bmatrix} \tilde{A} & \tilde{B}_i \\ \tilde{C}_j & \tilde{D}_{ji} \end{bmatrix}
\] (2.25)

It can be shown (see for example [27]) that the matrices \( \tilde{A}, \tilde{B}_i, \tilde{C}_j \) and \( \tilde{D}_{ji} \) can be obtained from the state-space representation of the system \( P_{ji}(z) \) in (2.18) as
\[
\tilde{A} = A_j^N
\] (2.26)
\[
\tilde{B}_i = \begin{bmatrix} A_j^{N-1}B_{jf} & A_j^{N-2}B_{jf} & \ldots & B_{jf} \end{bmatrix}
\] (2.27)
\[
\tilde{C}_j = \begin{bmatrix} C_{jf}^T (C_{jf}A_j)^T & \ldots & (C_{jf}A_j^{N-1})^T \end{bmatrix}^T
\] (2.28)
\[
\tilde{D}_{ji} = \begin{bmatrix} D_{ji} & 0 & \ldots & 0 \\ C_{jf}B_{jf} & D_{ji} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{jf}A_j^{N-2}B_{jf} & C_{jf}A_j^{N-3}B_{jf} & \ldots & D_{ji} \end{bmatrix}
\] (2.29)

Next, following the scheme in Figure 2.7 we can obtain the state-space representation of the lifted system \( P_{ji}(z) \) as follows
\[
P_{ji}(z) = \begin{bmatrix} A & B_i \\ C_j & D_{ji} \end{bmatrix} := \begin{bmatrix} \tilde{A} & \tilde{B}_i \tilde{H}_{ni} \\ \tilde{C}_j & \tilde{D}_{ji} \tilde{H}_{ni} \end{bmatrix}
\] (2.30)
where

\[ A = A_f^N \]  \hspace{2cm} (2.31) \\
\[ B_i = \left[ \sum_{l=0}^{n_i-1} A_f^{N-1-l} B_{if} \sum_{l=n_i}^{2n_i-1} A_f^{N-1-l} B_{if} \ldots \sum_{l=N-n_i}^{N-1} A_f^{N-1-l} B_{if} \right] \]  \hspace{2cm} (2.32) \\
\[ C_j = \left[ C_{jf}^T (C_{jf} A_f^{m_j})^T \ldots (C_{jf} A_f^{N-m_j})^T \right]^T \]  \hspace{2cm} (2.33) \\
\[ D_{ji} = \left[ D_{(r,s)} \right] \text{ with } r = 0, 1, \ldots, m_j - 1 \quad s = 0, 1, \ldots, n_i - 1 \]  \hspace{2cm} (2.34)

with

\[ D_{(r,s)} = D_{jif} \Psi_{[s_m, (s+1)n_i]}(rm_j) + \sum_{l=s_m}^{(s+1)n_i-1} C_{jf} A_f^{m_j-1-l} B_{if} \Psi_{[0,rm_j]}(l) \]  \hspace{2cm} (2.35)

and

\[ \Psi_{[r,s]}(l) := \begin{cases} 
I, & \text{if } r \leq l < s \\
0, & \text{otherwise.} 
\end{cases} \]  \hspace{2cm} (2.36)

Notice from the expressions (2.31)-(2.34) that the size of the inputs and outputs of \( P_{ji}(z) \) are \( n_i d_u \) and \( m_j d_y \) respectively. Indeed, if \( n_i = 1 \) and \( m_j = 1 \) for a given \( N \) it holds that \( P_{ji}(z) \equiv \tilde{P}_{ji}(z) \).

Consider as example the case when \( n_i = 2 \) and \( m_j = 3 \) thus \( N = 6 \), using (2.31) to (2.36) we obtain the state-space representation of \( P_{ji}(z) \) as

\[ A = A_f^6 \]  \hspace{2cm} (2.37) \\
\[ B_i = \left[ (A_f^5 + A_f^4) B_{if} \ (A_f^3 + A_f^2) B_{if} \ (A_f + I) B_{if} \right] \]  \hspace{2cm} (2.38) \\
\[ C_j = \left[ C_{jf}^T (C_{jf} A_f^3)^T \right]^T \]  \hspace{2cm} (2.39) \\
\[ D_{ji} = \begin{bmatrix} D_{jif} & 0 & 0 \\
C_{jf} A_f^2 B_{if} + C_{jf} A_f B_{if} & C_{jf} A_f B_{if} + D_{jif} & 0 \end{bmatrix} \]  \hspace{2cm} (2.40)

Notice that we obtain a lifted LTI system \( P_{ji}(z) \) that is also causal. Causality is an inherit property of all lifted LTI systems that are obtained from multirate systems using the methodology presented in this Section.
On the other hand, periodicity is also a valuable and useful property in multirate systems. As instance, it will be used for the computation of the frequency response matrix of a multirate system later in this section. As introduced and analyzed in [26], a multirate system \( P_{ji}(z) \) is a \((m_j, n_i)\)-shift varying operator. If \( n_i = m_j = N \) then the system \( P_{ji}(z) \) is a \( N \)-periodic system. Such operator also preserves the \( H_\infty \) norm of the system in which it operates.

We can compute an equivalent \( N \)-periodic system from the multirate system \( P_{ji}(z) \) if \( n_i \neq m_j \neq N \). Such system can be easily obtained from the fact that \( n_i \pi_i = m_j \pi_j = N \). Consider the operators \( H_{\pi_i} \) and \( S_{\pi_j} \) to have the same structure as defined in (2.20) and (2.20) but with \( \pi_i \) and \( \pi_j \) dimensions. We can obtain an augmented system \( P_{ji}^{aug}(z) \)

\[
P_{ji}^{aug}(z) := H_{\pi_i} P_{ji}(z) S_{\pi_j} \quad (2.41)
\]

In the case when \( n_i = m_j = N \) it yields that \( P_{ji}(z) \equiv P_{ji}^{aug}(z) \).

Following the same example as above (i.e. \( n_i = 2, m_j = 3, N = 6 \)) we obtain the state-space representation of the augmented system \( P_{ji}^{aug}(z) \)

\[
P_{ji}(z) = \begin{bmatrix} A_{ji}^{aug} & B_{ij}^{aug} \\ C_{ji}^{aug} & D_{ji}^{aug} \end{bmatrix} \quad (2.42)
\]

where

\[
A_{ji}^{aug} = A_f^6 \quad (2.43)
\]

\[
B_{ij}^{aug} = \begin{bmatrix} (A_f^5 + A_f^4)B_{ij} & 0 & (A_f^3 + A_f^2)B_{ij} & 0 & (A_f + I)B_{ij} & 0 \end{bmatrix} \quad (2.44)
\]

\[
C_{ji}^{aug} = \begin{bmatrix} C_{jf}^T & C_{jf}^T & (C_{jf}A_f^3)^T & (C_{jf}A_f^3)^T & (C_{jf}A_f^3)^T \end{bmatrix}^T \quad (2.45)
\]
\[
\begin{bmatrix}
D_{ji}^f & 0 & 0 & 0 & 0 \\
D_{ji}^f & 0 & 0 & 0 & 0 \\
D_{ji}^f & 0 & 0 & 0 & 0 \\
C_{jj}A_j^2B_{ij} + C_{jjf}A_jB_{ij} & 0 & C_{jjf}A_jB_{ij} + D_{ji}^f & 0 & 0 \\
C_{jj}A_j^2B_{ij} + C_{jjf}A_jB_{ij} & 0 & C_{jjf}A_jB_{ij} + D_{ji}^f & 0 & 0 \\
C_{jj}A_j^2B_{ij} + C_{jjf}A_jB_{ij} & 0 & C_{jjf}A_jB_{ij} + D_{ji}^f & 0 & 0
\end{bmatrix}
\]

Note that \( D_{ji}^a(z) \) is a causal system that has \( Nd_{ui} \) and \( Nd_{yj} \) outputs and it is now a \( N \)-periodic system.

### 2.3 Polyphase representation of multirate systems.

A slightly different approach for the representation of multirate discrete systems that is based in the polyphase representation is presented in this section. Such representation was first introduced in [47] and it is commonly used in digital signal processing since it makes a very useful tool for the design of filter banks. The interested reader about this topic is also referred to [2].

A transfer function \( P_{ji}(z) \) can be expressed in terms of its polyphase components \( P_l(z) \) as

\[
P_{ji}(z) = \sum_{i=0}^{N} z^l P_l(z^N)
\]

Note that in the polyphase components, the subindices \( j \) and \( i \) have been dropped for a simplification in the notation.

It can be shown [2] that the obtained lifted system \( \tilde{P}_{ji}(z) \) can be expressed also in terms of the polyphase components \( P_l(z) \) of the base rate \((1/T_f)\) system \( P_{ji}(z) \)
The representation in (2.48) is equivalent to the one given in (2.25)-(2.28) thus we can obtain the state-space representation of each of the polyphase components $P_l(z)$ as

$$
\tilde{P}_{ji}(z) = \begin{bmatrix}
P_0(z) & z^{-1}P_{N-1}(z) & \cdots & z^{-1}P_2(z) & z^{-1}P_1(z) \\
p_1(z) & P_0(z) & \cdots & z^{-1}P_3(z) & z^{-1}P_2(z) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
P_{N-2}(z) & P_{N-3}(z) & \cdots & P_0(z) & z^{-1}P_{N-1}(z) \\
P_{N-1}(z) & P_{N-2}(z) & \cdots & P_1(z) & P_0(z)
\end{bmatrix} \quad (2.48)
$$

The proof for the equivalences in (2.49)-(2.51) is straightforward and can be obtained by computing the transfer function for each system. More precisely, Assume that $A^{-1}$ exists and consider two systems $P_1(z)$ and $P_2(z)$ with state-space representation

$$
P_1(z) = \frac{A_f^N}{C_f} + \frac{A_f^{N-1}B_{if}}{C_fA_f^{k_0}} + \frac{A_f^{N-1-k_0}B_{if}}{D_{jf}} \equiv \begin{bmatrix} A_f^N & A_f^{N-1-k_0}B_{if} \\ C_jfA_f^{k_2} & D_{jf} \end{bmatrix} \quad (2.49)
$$

$$
P_2(z) = \frac{A_f^N}{C_f} + \frac{A_f^{N-1}B_{if}}{C_fA_f^{l+k_1}} + \frac{A_f^{N-1-k_1}B_{if}}{C_fA_f^{l+k_1}} \equiv \begin{bmatrix} A_f^N & A_f^{N-1-k_1}B_{if} \\ C_jfA_f^{l+k_2} & D_{jf} \end{bmatrix} \quad (2.50)
$$

$$
z^{-1}P_l(z) = \frac{A_f^N}{C_f} + \frac{A_f^{l-1}B_{if}}{C_fA_f^{k_2}} + \frac{A_f^{l-1-k_2}B_{if}}{D_{jf}} \equiv \begin{bmatrix} A_f^N & A_f^{l-1-k_2}B_{if} \\ C_jfA_f^{k_2} & D_{jf} \end{bmatrix} \quad (2.51)
$$

where

$$
k_0 = 0, \ldots , N-1. \quad (2.52)
$$

$$
l = 1, \ldots , N-1. \quad (2.53)
$$

$$
k_1 = 0, \ldots , N-1-l. \quad (2.54)
$$

$$
k_2 = 0, \ldots , l-1. \quad (2.55)
$$
Proof. Compute the transfer function of \( P_2(z) \) as

\[
P_2(z) = CA^k(zI - A^N)^{-1}A^{m-k}B + D \\
= C(A^{-k})^{-1}(zI - A^N)^{-1}A^{m-k}B + D \\
= C((zI - A^N)A^{-k})^{-1}A^{m-k}B + D \\
= C((zI - A^N)A^{-k})^{-1}(A^k)^{-1}A^mB + D \\
= C(A^k(zI - A^N)A^{-k})^{-1}A^mB + D \\
= C(zA^kA^{-k} - A^kA^NA^{-k})^{-1}A^mB + D \\
= C(zI - A^N)^{-1}A^mB + D \\
= P_1(z)
\]

Following the representation in Figure 2.7 we can compute the lifted system \( \bar{P}_{ji}(z) \) as \( \bar{P}_{ji}(z) = S_{mj}\tilde{P}_{ji}(z)H_{ni} \). We first compute the \( r \)th row of \( S_{mj}\tilde{P}_{ji}(z) \) as

\[
\left[ P_{rmj}(z) \ P_{rmj-1}(z) \ \cdots \ P_0(z) \ z^{-1}P_{N-1}(z) \ \cdots \ z^{-1}P_{rmj+2}(z) \ z^{-1}P_{rmj+1}(z) \right]
\]

(2.56)

Carrying out the product between expression (2.56) and the \( s \)th column of \( H_{ni} \) we obtain the system \( \bar{P}_{ji}(r, s) \). Such system corresponds to the \( r \)th row and \( s \)th column of the lifted system \( \bar{P}_{ji}(z) = S_{mj}\tilde{P}_{ji}(z)H_{ni} \) and its elements are based on the polyphase components of \( P_{ji}(z) \). Its expression is as follows

\[
\bar{P}_{ji}(r, s) = \sum_{l=sn_i}^{(s+1)n_i-1} P_{rmj-l}\Psi_{[0,rmj+1]}(l) + z^{-1}P_{N+rmj-l}\Psi_{[rmj+1,N]}(l) \tag{2.57}
\]

and \( \Psi_{[r,s]}(l) \) is defined in (2.36), with \( r = 0, 1, \ldots m_j - 1 \), and \( s = 0, 1, \ldots n_i - 1 \).

As an illustrative example, consider again a multirate system with \( n_i = 2, m_j = 3 \) and \( N = 6 \). The polyphase representation \( \bar{P}_{ji}(z) \) of such multirate system can be
obtained following the expression (2.57), therefore

\[ P_{ji}(z) = \begin{bmatrix} P_0(z) + z^{-1}P_5(z) & z^{-1}(P_4(z) + P_3(z)) & z^{-1}(P_2(z) + P_1(z)) \\ P_3(z) + P_2(z) & P_1(z) + P_0(z) & z^{-1}(P_5(z) + P_4(z)) \end{bmatrix} \] (2.58)

As computed in (2.42) we can compute the \( \hat{N} \)-periodic system \( P_{ji}^{\text{aug}}(z) \). The system obtained for this example is

\[ P_{ji}^{\text{aug}}(z) = \begin{bmatrix} P_0(z) + z^{-1}P_5(z) & 0 & z^{-1}(P_4(z) + P_3(z)) & 0 & z^{-1}(P_2(z) + P_1(z)) & 0 \\ P_0(z) + z^{-1}P_5(z) & 0 & z^{-1}(P_4(z) + P_3(z)) & 0 & z^{-1}(P_2(z) + P_1(z)) & 0 \\ P_0(z) + z^{-1}P_5(z) & 0 & z^{-1}(P_4(z) + P_3(z)) & 0 & z^{-1}(P_2(z) + P_1(z)) & 0 \\ P_3(z) + P_2(z) & 0 & P_1(z) + P_0(z) & 0 & z^{-1}(P_5(z) + P_4(z)) & 0 \\ P_3(z) + P_2(z) & 0 & P_1(z) + P_0(z) & 0 & z^{-1}(P_5(z) + P_4(z)) & 0 \\ P_3(z) + P_2(z) & 0 & P_1(z) + P_0(z) & 0 & z^{-1}(P_5(z) + P_4(z)) & 0 \end{bmatrix} \] (2.59)

### 2.4 Frequency response matrix of multirate discrete systems.

In an analogous manner that time-domain lifting allows a representation of a LPTV system as a LTI system, one can parse the frequency range \([0, 2\pi/T_f]\) (with \(T_f\) being the fast (base) sampling period) into \(N\) blocks \([2k\pi/T_s, 2(k+1)\pi/T_s], k = 0, \ldots, N-1\) and from the frequency response of a signal \(W(e^{jΩT_f}), Ω \in [0, 2\pi/T_f]\) build a multivariate frequency response as follows

\[ \hat{W}(e^{jΩT_f}) \equiv [W^T(e^{jΩT_f}), W^T(e^{j(Ω+\frac{2\pi}{T_s})T_f}), \ldots, W^T(e^{j(Ω+\frac{2\pi(N-1)}{T_s})T_f})]^T, Ω \in [0, 2\pi/T_s] \]

where \(T_s = NT_f\) is the slow sampling period. Then the frequency response matrix \(G_{FR}(e^{jΩT_f})\) relates such signals \(\hat{W}\) and \(\hat{V}\) for the input and output of the \(\hat{N}\)-periodic LPTV system in the usual manner:

\[ \hat{V}(e^{jΩT_f}) = G_{FR}(e^{jΩT_f})\hat{W}(e^{jΩT_f}), \ Ω \in [0, 2\pi/T_s]. \]
In the following, we briefly review how the frequency response matrix $G^{FR}(z)$ can be computed and used to quantify aliasing in the multirate systems considered here.

Consider the state space representation of a lifted system $H(z)$ relating the $z$-transforms of the lifted input and output signals is $W(z)$ and $V(z)$.

$$H(z) = \frac{V(z)}{W(z)} = C_c(zI - A_c)^{-1}B_c + D_c$$

(2.60)

Such system is N-periodic and has been obtained following the expressions presented in Section 2.2.

Next, starting with the $z$-transform $V(z) = \sum_{k=0}^{\infty} v(k)z^{-k}$ of a signal $v(k)$, the transform of a N-period sampled version of $v(k)$ is

$$V(z, i) = \sum_{m=0}^{\infty} v(i + mN)z^{-m}$$

(2.61)

and the transform of the lifted signal $V(k)$ can be expressed as

$$V(z) = \begin{bmatrix} V_T(z, 0) & V_T(z, 1) & \cdots & V_T(z, N - 1) \end{bmatrix}^T.$$  

(2.62)

$W(z)$ can be similarly expressed.

Then, it follows that

$$V(z) = \sum_{l=0}^{N-1} z^{-l}V(z^N, l)$$

(2.63)

and using (2.62) it yields

$$V(z) = \begin{bmatrix} I_{n_o} & z^{-1}I_{n_o} & \cdots & z^{-(N-1)}I_{n_o} \end{bmatrix} V(z^N).$$

(2.64)

Next, let $\phi = e^{\frac{2\pi j}{N}}$ and note that $\phi^{pN} = 1$, $p \in \mathbb{Z}$. Following a similar analysis in [48], we express (2.64) as

$$V(z\phi^p) = \begin{bmatrix} I_{n_o} & (z\phi^p)^{-1}I_{n_o} & \cdots & (z\phi^p)^{-(N-1)}I_{n_o} \end{bmatrix} V(z^N),$$

(2.65)
which leads to

\[
V(z\phi^p) = \begin{bmatrix}
I_{n_0} & \phi^{-p}I_{n_0} & \cdots & \phi^{-(N-1)}I_{n_0} \\
I_{n_0} & 0_{n_0} & \cdots & 0_{n_0} \\
0_{n_0} & z^{-1}I_{n_0} & \cdots & 0_{n_0} \\
\vdots & \vdots & \ddots & \vdots \\
0_{n_0} & 0_{n_0} & \cdots & z^{-(N-1)}I_{n_0}
\end{bmatrix}
D_{n_0}(z) V(z^N) \tag{2.66}
\]

where we define \( D_{n_0}(z) = \text{diag}\{I_{n_0}, z^{-1}I_{n_0}, \ldots, z^{-(N-1)}I_{n_0}\} \), thus

\[
V(z\phi^p) = \begin{bmatrix}
I_{n_0} & \phi^{-p}I_{n_0} & \cdots & \phi^{-(N-1)}I_{n_0} \\
I_{n_0} & \phi^{-1}I_{n_0} & \cdots & \phi^{-(N-1)}I_{n_0} \\
\vdots & \vdots & \ddots & \vdots \\
I_{n_0} & \phi^{-(N-1)}I_{n_0} & \cdots & \phi^{-(N-1)(N-1)}I_{n_0}
\end{bmatrix} \Phi_{n_0} D_{n_0}(z) V(z^N). \tag{2.67}
\]

Then, we compute

\[
\hat{V}(z) \equiv \begin{bmatrix}
V(z) \\
V(z\phi) \\
\vdots \\
V(z\phi^{N-1})
\end{bmatrix} = \begin{bmatrix}
I_{n_0} & I_{n_0} & \cdots & I_{n_0} \\
I_{n_0} & \phi^{-1}I_{n_0} & \cdots & \phi^{-(N-1)}I_{n_0} \\
\vdots & \vdots & \ddots & \vdots \\
I_{n_0} & \phi^{-(N-1)}I_{n_0} & \cdots & \phi^{-(N-1)(N-1)}I_{n_0}
\end{bmatrix} \Phi_{n_0} \left( D_{n_0}(z) V(z^N) \right). \tag{2.68}
\]

where \( \Phi_m \) is the Discrete Fourier Transform (DFT) matrix with elements

\[
\Phi_m(i, j) = \phi^{-ij}I_m \quad i, j = 0, 1\ldots (N-1).
\]

Similarly to (2.68), we have that

\[
\hat{W}(z) = \Phi_m D_n(z) W(z^N). \tag{2.69}
\]

Since \( \Phi_m^{-1} = \frac{1}{N} \Phi_m^* \), the system matrix \( \Phi_m D_m(z) \) has an inverse computed as

\[
(\Phi_m D_m(z))^{-1} = D_m^{-1}(z) \Phi_m^{-1} = \frac{1}{N} D_m^{-1}(z) \Phi_m^*. \tag{2.70}
\]
Therefore,
\[ W(z^N) = \frac{1}{N} D_{n_i}^{-1}(z) \Phi_{n_i}^* \tilde{W}(z). \]  
(2.71)

Since \( \tilde{V}(z) = H(z)W(z) \) follows from (2.60), we can use expressions (2.68)-(2.71) to compute
\[ \tilde{V}(z) = \Phi_{n_o} \cdot D_{n_o}(z) \tilde{V}(z^N) = \Phi_{n_o} D_{n_o}(z) H(z^N) W(z^N) = \]
\[ = \frac{1}{N} \Phi_{n_o} D_{n_o}(z) H(z^N) D_{n_i}^{-1}(z) \Phi_{n_i}^* \tilde{W}(z) \]  
(2.72)

from which we obtain the desired expression for the frequency response matrix
\[ \frac{G_{FR}(z)}{1} = \frac{1}{N} \Phi_{n_o} D_{n_o}(z) H(z^N) D_{n_i}^{-1}(z) \Phi_{n_i}^*. \]  
(2.73)

An alternative expression for \( \frac{G_{FR}(z)}{1} \) commonly found in the literature (see, for example [28]-[46]) can obtained by expanding (2.71) as follows.
\[ \tilde{W}(z^N) = \frac{1}{N} \left[ \sum_{p=0}^{N-1} W^T(z\phi^p) \cdot z \sum_{p=0}^{N-1} \phi^p W^T(z\phi^p) \cdots z^{N-1} \sum_{p=0}^{N-1} \phi^{(N-1)p} W^T(z\phi^p) \right]^T \]  
(2.74)

Substituting (2.74) in \( \tilde{V}(z^N) = H(z^N)W(z^N) \) and then in (2.64) gives
\[ V(z) = \frac{1}{N} \left[ \begin{array}{cc} \sum_{p=0}^{N-1} W^T(z\phi^p) \cdot z \sum_{p=0}^{N-1} \phi^p W^T(z\phi^p) \cdots z^{N-1} \sum_{p=0}^{N-1} \phi^{(N-1)p} W^T(z\phi^p) \end{array} \right] \]
\[ \cdot \left[ \begin{array}{c} U(z\phi^p) \\
\sum_{p=0}^{N-1} \phi^p U(z\phi^p) \\
\vdots \\
z^{N-1} \sum_{p=0}^{N-1} \phi^{(N-1)p} U(z\phi^p) \end{array} \right]. \]  
(2.75)

Finally expanding (2.75) and grouping common terms yields
\[ V(z) = \sum_{p=0}^{N-1} G_p(z) W(z\phi^p) \]  
(2.76)
where
\[
G_p(z) = \frac{1}{N} \sum_{l=0}^{N-1} \sum_{i=0}^{N-1} \phi^{lp} z^{i-l} H_{l,i}(z^N). \tag{2.77}
\]

Alternative proof for (2.77) can be found, for example, in [28]. We can also rewrite equation (2.76) as
\[
V(z) = G_0(z) U(z) + \sum_{p=1}^{N-1} G_p(z) U(z\phi^p). \tag{2.78}
\]

Equations (2.72) and (2.76) express the relationship between \(z\)-transform of the input and output sequences, so \(G_{FR}(z)\) can be considered as a generalized transfer matrix of the LPTV system. Analyzing (2.78) as a time invariant system, we can see that the output is a shaped and frequency shifted version of the input and the terms \(G_p(z), p > 0\) reflect aliasing due to decimation. We can rewrite (2.76) in a matrix form as follows
\[
\hat{V}(z) = \begin{bmatrix}
G_0(z) & G_1(z) & \cdots & G_{N-1}(z) \\
G_{N-1}(z\phi) & G_0(z\phi) & \cdots & G_{N-2}(z\phi) \\
\vdots & \vdots & \ddots & \vdots \\
G_1(z\phi^{N-1}) & G_2(z\phi^{N-1}) & \cdots & G_0(z\phi^{N-1})
\end{bmatrix} \hat{W}(z) \tag{2.79}
\]

where \(\hat{V}(z) = [V^T(z) \ V^T(z\phi) \ \cdots \ \ V^T(z\phi^{N-1})]^T\) and \(\hat{W}(z) = [W^T(z) \ W^T(z\phi) \ \cdots \ W^T(z\phi^{N-1})]^T\).

In [46], \(G_{FR}(z)\) is called the frequency response matrix of the LPTV system while in [2] and [48] is referred to as the Alias Component (AC) matrix. Two special cases simplify the computation of the frequency response matrix: for a fast rate \((T_f)\) LTI system, the matrix \(G_{FR}(z)\) is block diagonal and for a slow rate \((T_s = NT_f)\) LTI system, the matrix \(G_{FR}(z)\) has all its elements \(G_p(z)\) equal.
2.5 Analysis of multirate systems.

Multirate systems are $N$ periodic LPTV systems and as such they are affected by aliasing, i.e., there exists transfer of signal content from one frequency band to another between its input and its output. Aliasing in LPTV systems can be quantified in terms of the frequency response matrix that is a generalization of the frequency response concept for LTI systems. Moreover, the analysis of the aliasing in multirate systems is based on the computation of the distance of a multirate system to the subspace of LTI systems.

We can measure this distance using $l_2$-induced norms as in [46]. For an LPTV system $G$ with period $T_s$, its $l_2$-induced norm is defined as

$$
\|G\| := \sup\{\|v\|_2 : \|u\|_2 = 1\}.
$$

(2.80)

It is proven in [49] that the $l_2$-induced norm of an LPTV system $G$ equals the $\infty$-norm of its frequency response matrix, thus

$$
\|G\| = \|G^{FR}\|_\infty = \max_{\frac{-\pi}{T_s} \leq \omega \leq \frac{\pi}{T_s}} \sigma_{max}[G^{FR}(\omega)].
$$

(2.81)

Then the distance of a stable, LPTV system $G$ to the subspace of LTI systems is given by

$$
\mu_\infty := \inf_{LTI H} \|G - H\|.
$$

(2.82)

A minimizing LTI system $H_{opt}$ can be used as an LTI approximation to the LPTV system $G$ and the quantity $\mu_\infty$ can serve as a measure of the aliasing in the LPTV system $G$ [46].

The following are computable lower and upper bounds on $\mu_\infty$:

$$
\mu_\infty \geq \| \begin{bmatrix} G_1(z) & G_2(z) & \cdots & G_{N-1}(z) \end{bmatrix} \|_\infty,
$$

(2.83)
$$\mu_{\infty} \leq \left\| \begin{bmatrix} 0 & G_1(z) & \cdots & G_{N-1}(z) \\ G_{N-1}(z\phi) & 0 & \cdots & G_{N-2}(z\phi) \\ \vdots & \vdots & \ddots & \vdots \\ G_1(z\phi^{N-1}) & G_2(z\phi^{N-1}) & \cdots & 0 \end{bmatrix} \right\|_{\infty}. \tag{2.84}$$

In the special case $N = 2$, the lower and upper bound become equal to $\mu_{\infty} = \|G_1\|_{\infty}$ and $H_{opt} = G_0$.

### 2.6 Summary and Conclusions

We have introduced in this chapter the fundamentals for the representation and analysis of multirate systems. Three different representations are derived by using several results in the existing literature. Performance measures are also given for the analysis and comparison of multirate systems.

The following Chapters use these results for the design of multirate $H_\infty$ optimal controllers and the study of their properties and performance. Thus, we apply the fundamental results presented above for the design of $H_\infty$ optimal multirate controllers using a two-stage design approach and also to obtain causal multirate $H_\infty$ optimal controllers. Finally, we present a new approach for the design of filter banks.
Chapter 3

Two-Stage Design of Multirate $\mathcal{H}_\infty$ Optimal Controllers

Multirate sampled data systems employ more than one sampling frequency. Such situations may arise in practical control systems when, for example, sampling rates for measurement signals are constrained and sampling rates for control inputs can be increased to enhance performance. Another example of a multirate control system comes up when different system components such as actuators have different bandwidths and can be operated at different sampling rates thus reducing real time computation without essentially sacrificing performance.

The operation at different sample frequencies requires the introduction in the system of downsamplers to reduce the sampling rate and upsamplers to increase it. Sampling some of the signals at a slower rate in a multirate system introduces aliasing that may cause loss of essential information. On the other hand, it also could lead to control signals with intersample ripple. In [32] and [50] such problem is addressed by forcing the control inputs to converge to a steady state. Indeed, intersample behavior becomes smaller for a faster sampling period.
The previously mentioned representations of multirate systems can be effectively used for assessing the loss of performance due to aliasing. These results have been also successfully used for multirate controller design within different control frameworks, such as as pole placement, LQG/LQR, and optimal $H_2$ and $H_\infty$ control (see for example [51] and references therein).

Some work in recent years has focused in approximately implementing single (fast) rate controllers by multirate ones with objective to reduce computational load in demanding applications such as hard disk drive servoing while maintaining performance levels [35], [37], and [38]. The controller is obtained by using any of the techniques mentioned above and then is decomposed in fast and slow modes.

In [37] and [38] the slow rate part of the controller is implemented by using interlacing to make the computations uniform over the extent of the slow time period. More recent work [39] proposes a scheme using interlacing by designing the slow rate controller so that its output matches the fast rate controller at slow sample instances.

In all these methods, there is only one synthesis step in which the fast rate controller is designed and then it is modified to a multirate system to reduce the computational load during real-time implementation. This controller transformation is carried out in an open-loop fashion without regard to the plant and specifications at hand. The multirate design is usually checked after the fact for excessive aliasing in the system due to the slow rate resampling. Indeed, this problem can be eased by including notch filters in the design process to minimize aliasing. However, this solution needs to be tailored specifically for each system, it is dependent on the sampling times used, and the use of notch filters designed separately may affect the optimality of the resulting design.

In this chapter, we propose a two-stage multirate controller design procedure. In the first stage, a single (fast) rate controller is designed by solving an appropriately
formulated $H_\infty$ optimal control problem; it is assumed that the fast rate controller obtained meets all design requirements. Next, the fast rate controller is decomposed into low frequency and high frequency parts. Then in the second stage, the high frequency part of the controller is retained and, unlike with other approaches, the low frequency part is *redesigned* at the slow rate using $H_\infty$ optimal control and a lifting technique [27]. Thus the proposed approach designs controllers both at fast and slow rates instead of merely resampling the low frequency modes of the fast rate controller. Lifting transfers information in the design from fast rate to slow rate so that no unnecessary performance loss occurs due to downsampling. Most importantly, the slow rate redesign takes into account the possibility of aliasing in the system and can compensate for it.

The remainder of this Chapter is organized as follows. In Section 3.1, we review some key results on multirate systems using lifting and frequency domain representations particularly applied to the proposed design approach. In Section 3.2, we describe the proposed algorithm in more detail and discuss some of its characteristics. In addition, the steps for the controller decomposition are presented. Two design examples and comparison with some previous methods are presented in Section 3.3. Finally, conclusions are given in Section 3.4.

### 3.1 Background and Preliminary Results in Multirate Systems

In this section, we review the background results on the lifting technique used to design the slow rate controller in our approach and on the frequency domain analysis of multirate systems.
3.1.1 Time-Domain Lifting

Consider a discrete time system $P(z)$ with inputs $w$ and $u_f$, outputs $v$ and $y_f$ of respective dimensions $n_i$, $n_c$, $n_o$, and $n_m$. These signals are assumed to be operating at the fast rate $1/T_f$. We also assume that $u_f$ results by up-sampling and holding a slow rate signal $u$ and that $y_f$ is down-sampled to a slow rate signal $y$; thus $u$ and $y$ operate at the slow rate $1/T_s$, where $T_s = NT_f$. Such a system arises by sampling a continuous time plant $G(s)$ at the fast rate but subsequently down-sampling the measurement $y$ and up-sampling the control $u$ signals so that a slow-rate controller can be used.

Figure 3.1 shows the lifted system representation for the computation of the LTI system $P(z)$ operating at the slow rate $1/T_s$. Note that this is a particular case (i.e., $p=q=2$) of the system presented in Section 2.2.

Assuming a minimal state-space realization for $P(z)$:

$$ P(z) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} A_f & B_f \\ C_f & D_f \end{bmatrix} = \begin{bmatrix} A_f & B_{1f} & B_{2f} \\ C_{1f} & D_{11f} & D_{12f} \\ C_{2f} & D_{21f} & D_{22f} \end{bmatrix}, \quad (3.1) $$

and using expressions (2.31)-(2.34) with $n_1 = m_1 = 1$ and $n_2 = m_2 = N$, we obtain
the lifted system $P(z)$ which has the following state-space realization:

$$
P = \begin{bmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{bmatrix} = \begin{bmatrix}
A & B_1 & B_2 \\
C_1 & D_{11} & D_{12} \\
C_2 & D_{21} & D_{22}
\end{bmatrix}
$$  \hspace{1cm} (3.2)

where the matrices in (3.2) are given by

$$
A = A_f^N \hspace{1cm} (3.3)
$$

$$
B_1 = \begin{bmatrix}
A_f^{N-1}B_1f & A_f^{N-2}B_1f & \cdots & B_1f
\end{bmatrix} \hspace{1cm} (3.4)
$$

$$
B_2 = \begin{bmatrix}
(A_f^{N-1} + A_f^{N-2} + \cdots + A_f + I)B_2f
\end{bmatrix} \hspace{1cm} (3.5)
$$

$$
C_1 = \begin{bmatrix}
C_{1f}^T & (C_{1f}A_f)^T & \cdots & (C_{1f}A_f^{N-1})^T
\end{bmatrix}^T \hspace{1cm} (3.6)
$$

$$
C_2 = C_{2f} \hspace{1cm} (3.7)
$$

$$
D_{11} = \begin{bmatrix}
D_{11f} & 0 & \cdots & 0 \\
C_{1f}B_1f & D_{11f} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{1f}A_f^{N-2}B_1f & C_{1f}A_f^{N-3}B_1f & \cdots & D_{11f}
\end{bmatrix} \hspace{1cm} (3.8)
$$

$$
D_{12} = \begin{bmatrix}
D_{12f} \\
C_{1f}B_2f + D_{12f} \\
\vdots \\
C_{1f}A_f^{N-2}B_2f + \cdots + D_{12f}
\end{bmatrix} \hspace{1cm} (3.9)
$$

$$
D_{21} = \begin{bmatrix}
D_{21f} & 0 & \cdots & 0
\end{bmatrix} \hspace{1cm} (3.10)
$$

$$
D_{22} = D_{22f} \hspace{1cm} (3.11)
$$

### 3.1.2 Frequency response matrix of multirate systems

In the following, we briefly review how the frequency response matrix $G^{FR}(z)$ obtained in Section 2.4 can be used to quantify aliasing in the multirate systems considered in the proposed design approach. More specifically, we consider the series
and parallel multirate controller configurations depicted in Figure 3.2. By absorbing the fast rate controller \( K_{HF}^S(z) \) or \( K_{HF}^P(z) \) into the open loop plant \( P(z) \), lifting the extended plant using the state-space formulas of Section 3.1.1, and closing the loop with the slow rate controller \( K_j^S(z) \) or \( K_j^P(z) \) (\( j = RE, 2S \)), we obtain the appropriate closed loop lifted system

\[
H(z) = \frac{V(z)}{W(z)} = C_s(zI - A_s)^{-1}B_s + D_s \tag{3.12}
\]

relating the z-transforms of the lifted input and output signals \( W(z) \) and \( V(z) \).

Figure 3.2: Control system with multirate controller using a) Series and b) Parallel configuration with \( j = RE, 2S \) for the resampled and two-stage controller respectively.

Note that we will obtain four different lifted closed loops (i.e., \( H_{2S}^S(z) \), \( H_{2S}^P(z) \), \( H_{RE}^S(z) \), and \( H_{RE}^P(z) \)).

Finally, using the expression (2.73) derived in Section 2.4, we can compute the corresponding frequency response matrix for each of the four lifted closed loop systems.

The frequency response matrix is used in two ways: first, to quantify the performance of multirate designs, and secondly, to measure aliasing in the multirate closed loop system.
### 3.2 Two Stage Design

#### 3.2.1 Proposed Design Approach

The multirate controller design proposed in this Chapter is obtained via the following steps:

1. **Controller design at fast rate.** An optimal $\mathcal{H}_\infty$ controller is designed at the fast sampling rate solving an appropriately formulated $\mathcal{H}_\infty$ optimal control problem. It is assumed that this controller addresses all design requirements in a satisfactory manner; the issue is to approximate this controller by a multirate system so that the fast rate part of the multirate controller fits within the required “budget” of fast rate computations.

2. **Controller decomposition in high and low frequency parts.** The fast rate controller obtained previously is decomposed in high and low frequency parts using either a series or a parallel configuration (see Section 3.2.2 for more details).

3. **Lifting of the extended plant.** The high frequency part of the fast rate controller is incorporated into the plant of the $\mathcal{H}_\infty$ design in the first step and then the new plant is lifted to slow rate (using the procedure discussed in Section 3.1.1) obtaining a extended plant. The use of lifting allows the formulation of all specifications into slow rate without unnecessary information loss.

4. **Design of slow rate controller.** A slow rate optimal $\mathcal{H}_\infty$ controller is designed based on the extended plant. The final multirate controller consists of the high frequency part of the fast rate controller designed in the first step and the slow rate controller designed in this step.
3.2.2 Parallel and Series Decomposition

In Step 2 of the previous procedure the fast rate controller $K_{FR}$ is decomposed in either a series configuration

$$K_{FR} = K_{LF}^{S} \cdot K_{HF}^{S}$$

(3.13)

or a parallel configuration

$$K_{FR} = K_{LF}^{P} + K_{HF}^{P},$$

(3.14)

where the subsystems $K_{LF}^{S}$ and $K_{HF}^{S}$, or $K_{LF}^{P}$ and $K_{HF}^{P}$ contain the low frequency (slow) and high frequency (fast) modes of $K_{FR}$ respectively for the two configurations. In the series configuration, we require that the zeros of $K_{LF}^{S}$ and $K_{HF}^{S}$ are also (approximately) partitioned in low and high frequency zeros respectively.

More specifically, let $T_{f}$ be the fast sampling period Nyquist frequency $\omega_{NF} = \frac{\pi}{T_{f}}$ in rad/sec. For a multirate ratio $N$, the slow sampling time is $T_{s} = NT_{f}$ with Nyquist frequency $\omega_{Ns} = \frac{\omega_{NF}}{N}$. For systems resulting from sampling, a continuous time pole $-\zeta\omega_{n} + j\omega_{n}\sqrt{1 - \zeta^{2}}$ becomes a discrete time pole $re^{j\theta}$ with $r = e^{-\zeta\omega_{n}T_{f}}$ and $\theta = \omega_{n}T_{f}\sqrt{1 - \zeta^{2}}$. Then, a complex point $re^{j\theta}$ ($-\pi \leq \theta \leq \pi$ in radians), signifying a pole or a zero, is considered to be low frequency if

$$|\log(r) + j\theta| < \omega_{Ns}T_{f} \equiv \frac{\pi}{N},$$

(3.15)

which is equivalent with $\omega_{n} < \omega_{Ns}$; otherwise, it is considered high frequency. Figure 3.3 shows the condition (3.15) in the pole-zero map in discrete time. Notice that (3.15) applies to both real and complex poles or zeros with a real pole or zero characterized from $\theta = k\pi, k = 0, \pm 1$. Also note that negative real poles or zeros are high frequency.

Next, we give state-space algorithms to accomplish the decompositions (3.13) and (3.14). We remark that the algorithms are applicable for MIMO systems that can be unstable.
Series Decomposition

In series decomposition, we factor $K_{FR}(z) = K_{LF}^S(z) \cdot K_{HF}^S(z)$ where $K_{HF}^S(z)$ is biproper, the low and high frequency poles are separated accordingly as expressed above and their zeros satisfy (approximately) the same factorization. We refer to such factorization as the Series (Low-High) Frequency Factorization (SFF) of $K_{FR}(z)$.

The series decomposition can be accomplished via the following algorithm that is an adaptation of the canonical factorization algorithm of [52] as it is described in [8]. Let $K_{FR}(z) = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a minimal realization of $K_{FR}(z)$. We assume for the following algorithm $K_{FR}$ to be square and such that $D$ is nonsingular. Fortunately, the first assumption is usually met in practice while the latter can be relaxed as it is discussed in subsequent remarks; it is invoked here to simplify the presentation. Then, the low frequency poles and zeros of $K_{FR}$ are the eigenvalues of $A$ and $A - BD^{-1}C$ respectively that satisfy (3.15) and the high frequency poles and zeros are the eigenvalues that fail this condition.
Step 1

With \( n, n_l, m_h \) being the state dimension of \( K_{FR} \), the number of low frequency poles, and the number of high frequency zeros of \( K_{FR} \) respectively, let \( T_p \) be an \( n \times n_l \) matrix such that its columns span the eigenspace of \( A \) corresponding to the low frequency poles and \( T_z \) be an \( n \times m_h \) matrix such that its columns span the eigenspace of \( A - BD^{-1}C \) corresponding to the high frequency zeros. Matrices \( T_p \) and \( T_z \) can be found from the Schur factorization of \( A \) and \( A - BD^{-1}C \) [8]. If \( n_l + m_h = n \) and \( T = [T_z \ T_p] \) is nonsingular, go to Step 2, otherwise continue with Step 1a.

Step 1a

In this step, we need to (slightly) modify the zeros resulting from condition (3.15) in Step 1 because otherwise \( K_{HF}^S(z) \) cannot be biproper. This is accomplished by ordering the eigenvalues of \( A - BD^{-1}C \) in decreasing magnitude and taking the first \( n - n_l \) ones such that complex conjugate pairs are always included. Then, \( T_z \) is redefined accordingly. The complementarity condition that \( T \) is nonsingular is typically satisfied, otherwise some further modification in what zeros are selected for the high-frequency factor is required.

Step 2

Introduce the partitions

\[
T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix},
\]

(3.16)

where \( A_1, A_4 \) are of dimension \( m_h \) and \( n_l \) respectively. Then

\[
K_{LF}^S(z) = \begin{bmatrix} A_4 & B_2 \\ C_2 & D \end{bmatrix}, \quad K_{HF}^S(z) = \begin{bmatrix} A_1 & B_1 \\ D^{-1}C_1 & I \end{bmatrix}.
\]

(3.17)

Remarks: If \( D \) is singular, \( K_{FR} \) may have \( 0 \leq n_z \leq n \) finite zeros and \( n - n_z \) zeros
at infinity. In such case, $T_z = [T_{zf}, T_{z\infty}]$, where $T_{zf}$ satisfies
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
T_{zf} \\
U_f
\end{bmatrix}
= \begin{bmatrix}
T_{zf} \\
U_f
\end{bmatrix} Z_{HF}
\tag{3.18}
\]
for some $U_f$ and $Z_{HF}$ a diagonal matrix of selected high-frequency zeros of $K_{FR}$ and $T_{z\infty}$ satisfies
\[
\begin{bmatrix}
\tilde{A} - \alpha I & \tilde{B} \\
\tilde{C} & \tilde{D}
\end{bmatrix}
\begin{bmatrix}
T_{z\infty} \\
U_\infty
\end{bmatrix}
= 0
\tag{3.19}
\]
for some $U_\infty$ and
\[
\begin{align*}
\tilde{A} &= (A + \alpha I)^{-1}(\alpha A + I) \\
\tilde{B} &= (\alpha^2 - 1)(A + \alpha I)^{-1}B \\
\tilde{C} &= C(A + \alpha I)^{-1} \\
\tilde{D} &= D - C(A + \alpha I)^{-1}B
\end{align*}
\tag{3.20-3.23}
\]
are obtained by transforming the $z$-plane by $z = \frac{1-\alpha w}{w-\alpha}$ for some $0 < \alpha \leq 1$ such that $\tilde{D}$ is nonsingular. Thus, $T_{zf}$ and $T_{z\infty}$ are obtained from vectors spanning eigenspaces corresponding to the selected high-frequency and zeros at infinity respectively.

**Parallel Decomposition**

The parallel decomposition can be accomplished as follows.

**Step 1**

Let $T$ an orthogonal transformation such that
\[
T^T AT = \begin{bmatrix}
A_1 & A_2 \\
0 & A_4
\end{bmatrix}
\tag{3.24}
\]
is in Schur canonical form with the eigenvalues of $A_1$ comprising the low frequency modes and the eigenvalues of $A_4$ the high frequency modes of $K_{FR}(z)$. 

45
Step 2

Let \( T^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \) and \( CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \) and \( X \) be a solution of the Sylvester equation \( A_1 X - X A_4 = A_2 \). Then

\[
K_P(z) = \frac{A_1}{C_1} \begin{bmatrix} B_1 - X B_2 \\ 0 \end{bmatrix}, \quad K_H(z) = \begin{bmatrix} A_4 \\ C_1 X + C_2 \end{bmatrix} \begin{bmatrix} B_2 \\ D \end{bmatrix}.
\]

(3.25)

### 3.2.3 Analysis and properties of the designed controllers

The analysis of the multirate controllers designed using the proposed two-stage methodology is based on computing the distance of a multirate system to the subspace of LTI systems as presented in Section 2.5.

Since the (fast rate) LTI controller \( K_{FR} \) designed in Step 1 of the design procedure is best in the sense of the \( \mathcal{H}_\infty \) -norm performance criterion, we want to obtain a multirate controller that is similar to \( K_{FR} \). Here we measure similarity in terms of the distance of the multirate closed-loop to the fast rate closed-loop (\( CL_{FR} \)). As it was reviewed earlier in Section 2.5, we can compute this distance using (2.82), as the infinity norm of the Frequency Response Matrices of the multirate closed-loop system and \( CL_{FR} \) (viewed as a trivial multirate system). The two-stage design approach will always produce a better controller in the sense of \( \mathcal{H}_\infty \) -norm than the resampling approach, since both approaches use the same fast rate controller while the slow rate controller is optimally redesigned in the two stage approach. In particular, resampled designs can be unstable, while the redesign approach guarantees closed-loop stability.

In addition to the performance measurement, we also evaluate the computational load of a multirate controller. Let \( C_h \) and \( C_l \) be the computational load associated to a fast and slow rate pole respectively. We can obtain the computational cost of a multirate system as \( C_{MR} = C_h n_h + C_l n_l \) being \( n_h \) and \( n_l \) the number of fast and
slow rate poles of the multirate system respectively. Based on the decompositions
presented in Section 3.2.2, for a fast rate system we can find \( n_l \) and \( n_h \) such that
the condition (3.15) is satisfied or not for a fixed \( \omega_{N_s} \).

Defining \( C_h = C_0 \) as the base rate computational load and having \( N \) as the multi-
rate ratio (i.e. \( N = T_s/T_f \)), we know that \( C_l = C_0/N \). Thus the total computational
load of a multirate system can be expressed as follows:

\[
C_{MR} = C_h n_h + C_l n_l = C_0 n_h + \frac{C_0 n_l}{N} = C_0 (n_h + \frac{n_l}{N}).
\]  

(3.26)

Notice that for a fast rate system (i.e. \( N=1 \)) \( C_{FR} = C_0 (n_h + n_l) \) and as the
multirate ratio \( N \) becomes large \( C_{MR} \approx C_0 n_h \). We compare the complexity of the
multirate systems using the computational load reduction, which we define as the
percentage of computational load of the multirate controller over the computational
load of fast rate system. Thus

\[
\tilde{C}_j := 100 \times \frac{C_j}{C_{FR}}, \quad j = 2S, \text{ RE.}
\]  

(3.27)

where \( C_j \) is computed as expressed in (3.26). Combining (3.26) and (3.27) we obtain
an expression that does not depend on \( C_0 \),

\[
\tilde{C}_j := 100 \times \frac{n_h + n_l^2/N}{n_h + n_l^2}, \quad j = 2S, \text{ RE.}
\]  

(3.28)

The computational load reduction (3.28) express the percentage ratio between the
pole complexity of a multirate system and a single (fast) rate system. In the case of
study, the value of \( n_h \) in (3.28) is obtained in the presented controller decomposition,
thus it will remain equal for both the resampled and the 2-stage approaches. Notice
also that \( n_l^{RE} \equiv n_l \) while \( n_l^{2S} \geq n_l \).

The multirate controller proposed here has usually more modes than the multi-
rate controller designed by resampling. However, the additional modes are included
at the slow rate part of the controller and are in number no more than twice the
modes of the fast part of the controller which is common to our method and the design by resampling. Indeed, we have found that the inclusion of the high frequency part of the first controller, allows to effectively model-reduce the plant before the second controller design, therefore achieving in practice controller complexity similar to the resampled multirate controller. Since the fast rate controller is usually of low order and the additional modes operate at the slow rate, the modest additional computational load (if any at all) seems to be a reasonable trade-off for the increased performance.

3.3 Design Examples

We illustrate the proposed design approach with two design examples. In each design, we compare the performance of five controllers: the fast rate controller \(K_{FR}\) and four multirate controllers obtained using either the series or the parallel decomposition depicted in Figure 3.2 and either the two-stage or resampled approach. The two-stage (i.e. \(j = 2S\)) controllers are obtained by applying the proposed design approach presented in Section 3.2.1 while the resampled controllers (i.e. \(j = RE\)) are obtained by downsampling the low frequency part resulting from the series or parallel decomposition presented in Section 3.2.2. In [37] and [38], multirate controllers designed by resampling are interlaced. Interlacing is an interesting idea that allows to evenly distribute real-time computational load. Computation load is proportional to the number of modes and the rate at which they operate. If desired, the multirate controllers designed by our approach can also be interlaced in the same manner with resampling controllers. Here, our focus is how performance is best maintained with a multirate controller and, therefore, we do not interlace controllers to simplify the presentation. However, interlacing the slow rate controller can lead to (typically slightly) worse performance due to the addition of aliasing elements in (2.77).
3.3.1 SISO multirate controller design

The bode plot of the plant used for the SISO multirate controller design is shown in Figure 3.4. The plant has the typical shape of the frequency response of a Voice Coil Motor (VCM) in a Dual Stage Hard Disk Drive [38], [53]. The open-loop plant has four modes and the design objective is to robustly reject impulse disturbances at the measurement. The fast sampling rate \(1/T_f\) used is 50kHz and we have a multirate ratio of \(N = 3\) (i.e. the slow sampling rate is \(1/3T_f = 1/T_s\)). The plant transfer function (discretized at fast sampling rate) is given in (3.29).

\[
P(z) = -2.0178 \times 10^{-5} \cdot \frac{(z + 16.87)(z + 2.336)(z + 0.805)(z + 0.376)}{(z^2 - 1.982z + 0.982)(z^2 - 1.471z + 0.978)} \cdot \frac{(z + 0.06)(z^2 - 0.779z + 0.946)}{(z^2 - 1.006z + 0.963)(z^2 - 0.237z + 0.865)}
\]

(3.29)

We evaluate closed-loop performance of each controller by measuring the distance between the fast rate closed loop system and the multirate closed loop systems. We measure this distance in terms of \(H_\infty\) norm as in (2.81) and expressed as

\[
\mu_{CL}^{\infty} := \|CL_{FR} - CL_{lj}\|_\infty, \ l = S, P, \ j = 2S, \ RE.
\]

(3.30)
Table 3.1: $\mu^C_{\infty}$ for the SISO multirate controllers

<table>
<thead>
<tr>
<th></th>
<th>$\mu^C_{\infty}$</th>
<th>$\mu_{\infty} \geq$</th>
<th>$\mu_{\infty} \leq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CL_{2S}$</td>
<td>1.5943</td>
<td>1.4746</td>
<td>1.4747</td>
</tr>
<tr>
<td>$CL_{2S}^P$</td>
<td>3.5759</td>
<td>2.1774</td>
<td>2.1774</td>
</tr>
<tr>
<td>$CL_{RE}^S$</td>
<td>4.3494</td>
<td>4.0374</td>
<td>4.0382</td>
</tr>
<tr>
<td>$CL_{RE}^P$</td>
<td>24.9150</td>
<td>24.8936</td>
<td>24.9007</td>
</tr>
</tbody>
</table>

where $CL_{FR}$ is the weighted closed loop using the fast rate controller and $CL_j$ is the weighted closed loop for each of the multirate controllers considered. The closed-loop distances are shown in Table 3.1 and they are smaller for the multirate controllers based on redesign than for the multirate controllers based on resampling of the low frequency part. Indeed, redesigning at slow rate compensates for possible aliasing resulting from modes that the high frequency part of the fast rate controller did not attenuate.

We also plot $\sigma_{\max}(\omega)$ in (2.81), the maximum of which gives the similarity measure in (3.30) over the length of one period $0 \leq \omega \leq \frac{\pi}{T_s}$. We can see in Figure 3.5 that the closed loop corresponding to the two-stage series controller ($CL_{2S}^S$) is closest to the fast rate closed loop $CL_{FR}$ over most frequencies. It is clear that the parallel decomposition is less successful in eliminating aliasing than the series configuration. This observation can be explained since the fast rate series controller acts as an antialiasing filter that is automatically designed in Step 1 and Step 2 of the method.

We measure the aliasing content of the closed loop computing the distance ($\mu_{\infty}$) from the multirate (LPTV) system to the closest LTI system as in (2.82). Since the multirate ratio $N = 3$, we compute the lower (2.83) and upper bounds (2.84) and present them in Table 3.1. We observe that the bounds are tight and show that aliasing in the case of two stage controllers is smaller than the resampled controllers.

We found that the two-stage design controllers can be effectively model-reduced
Figure 3.5: \( \sigma_{\text{max}} \) plot for the weighted closed loop SISO systems without loss of performance. The high frequency part of the fast rate controller has four modes using both (series and parallel) decompositions and the slow rate controllers obtained have 15 and 16 modes for the series and parallel configuration, respectively. After model-reduction by using balanced model truncation, all four multirate controllers compared above have four fast and six slow modes and therefore identical computational characteristics.

The time responses of the closed loop systems (with using the model-reduced controllers as applicable) to an impulse disturbance applied at the measurement are compared in Figure 3.6. It can be seen that the fast rate controller \( K_{FR} \) gives the best time response that serves as our benchmark for evaluating performance in the time domain. We again see that the multirate controller using series decomposition \( K_{2S}^S \) has a performance similar to the fast rate controller. In both cases (parallel and series configuration) the controllers proposed in this Chapter result in a better time response than the ones using resampling of the low frequency modes at slow rate.

Finally, we also evaluate the trade-off between loss of performance and computational load reduction (3.28) as we increase the multirate ratio \( N \) for the series
configuration. The fast rate controller decomposition results in four high and six low frequency poles. The performance loss measure used for comparison is defined as a normalized percentage of the distance between the fast rate and the multirate closed loops (3.30) with respect to the closed loop fast rate $\mathcal{H}_\infty$ norm as follows:

$$\tilde{\mu}_{\infty}^{CL} := 100 \times \frac{\mu_{\infty}^{CL}}{\|CL_{FR}\|_\infty}. \quad (3.31)$$

The computational load and performance for the multirate controllers using the both configurations (series and parallel) comparing both resampled and two-stage approaches for the SISO controller design are shown in Figure 3.7 and Figure 3.7 respectively. We can see that for both configurations the performance loss is smaller when using the two-stage approach for all values of the multirate ratio $N$. Notice also that the controller obtained using the two-stage design the approach has larger computational load than the resampled approach. However, after model reduction of the slow rate part of the multirate controller, we obtain identical computation loads.
3.3.2 MIMO multirate controller design

The plant used for the MIMO multirate controller design is a dual stage Hard Disk Drive. The plant is taken from [33], but it has been model reduced to include the most significant modes. The fast sampling rate (1/T_f) used is 25kHz and we have
We consider in this section a track-seek problem similar to the one in [33] and design together feedforward and feedback controllers for both the VCM and PZT subsystems. More specifically, we take as the controller inputs the reference signal and the position error, and we take as the controller outputs the control signals for the VCM and PZT subsystems; thus we obtain a two-input two-output fast rate controller.

As in the SISO design example presented above, we evaluate closed loop perfor-
Table 3.2: $\mu_{\infty}^{CL}$ for the MIMO multirate controllers

<table>
<thead>
<tr>
<th></th>
<th>$\mu_{\infty}^{CL}$</th>
<th>$\mu_{\infty}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CL_{2S}^S$</td>
<td>2,631.07</td>
<td>1,515.10</td>
</tr>
<tr>
<td>$CL_{2S}^P$</td>
<td>2,753.43</td>
<td>1,964.86</td>
</tr>
<tr>
<td>$CL_{RE}^S$</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>$CL_{RE}^P$</td>
<td>10,362.61</td>
<td>10,362.49</td>
</tr>
</tbody>
</table>

† The closed loop becomes unstable with the series resampled multirate controller.

formance by using (3.30) to measure distances between closed loops. The results are presented in Table 3.2 and show that the controllers obtained with the two stage design approach are closer to the fast rate closed loop than the resampled controllers. Notice that the resampled controller in series configuration results in an unstable closed loop.

We also measure the aliasing content of the closed loop computing the distance ($\mu_{\infty}$). In this example, the multirate ratio $N = 2$, therefore, the lower and upper bound are equal and we can compute $\mu_{\infty} = \|G_1\|_{\infty}$ where $G_1$ is an element of the frequency response matrix (2.79). We can see in Table 3.2 that the values for the two stage controllers are smaller than the resampled controllers.

The norms shown at table 3.2 are obtained at $\omega = 0$ Hz as we can see in Figure 3.10, which plots $\sigma_{\text{max}}$ in (2.81) as a function of frequency over the length of one period $0 \leq \omega \leq \frac{\pi}{Ts}$. Again, the closed loop corresponding to the two-stage series controller ($CL_{2S}^S$) is the closest to the fast rate closed loop $CL_{FR}$ over all frequencies. A plot for the resampled series controller ($CL_{RE}^S$) is not given since its closed-loop is unstable.

The fast rate controller $K_{FR}$ has 14 (fast rate) modes that are split into 5 high frequency and 9 low frequency modes. The high frequency modes are common for all multirate controllers. Downsampling does not increase the order of the low rate controller, thus all resampled low rate controllers have 9 modes. In the two-stage
approach, the low rate controllers are redesigned; after model reduction their order becomes 10 for both the series and parallel configuration. We can see again that both resampled and two-stage redesigned controllers have about the same complexity while the performance is significantly better for the two stage design approach being very close to that obtained with the fast rate controller both with respect to frequency and time domain measures.

The time responses for the track seek task are plotted in Figure 3.11 for each of the closed loop systems (and for the model reduced controllers as applicable). It can be seen that the two-stage series multirate controller $K_{2S}^S$ has performance closest to that of the fast rate design. Notice that the resampled controller using series configuration $K_{RE}^S$ cannot compensate the system and its closed loop becomes unstable.

We also compute and compare the trade-off between computational load reduction (3.28) and performance loss (3.31) for the MIMO design example using the series configuration multirate controller. The decomposition used is the same as obtained
Figure 3.11: Time responses of the closed loop system using the 5 controllers designed for track seek.

above. The results obtained are shown in Figure 3.12 and Figure 3.13 for the series and parallel controller decomposition respectively. Notice that the performance loss is not shown for the resampled controller when using the series decomposition since the closed loop is unstable for all values of the multirate ratios $N$. However, the computational load reduction is show for comparison with the model reduced controller obtained using the two stage multirate controller design. We can see that the performance loss is 60% for $N = 2$ when using the two-stage approach (after model reduction) with a computational load reduction of 64%. Greater values of $N$ lead to values of $\tilde{\mu}_{CL}^C$ greater that 100%, which is not desirable. Although for this example the performance loss is greater than the one in the SISO example presented above, we found that for a similar computational load the two stage multirate controller is able to achieve closed loop stability while the resampled controller is unable to do so.
Figure 3.12: Performance and computational load plots for the MIMO design example using the series controller decomposition.

Figure 3.13: Performance and computational load plots for the MIMO design example using the parallel controller decomposition.

3.4 Conclusions

In this Chapter, a new methodology is presented for the design of $\mathcal{H}_\infty$ optimal multirate controllers. These controllers become multirate because the low frequency modes are implemented at a slower rate. A key feature of the proposed approach
is the redesign of the slow rate part of the controller rather than simply using resampling. The slow rate design is effected on the plant modified by the high frequency part of the controller operating at the fast sampling rate and by using lifting. This approach addresses potential problems of aliasing due to the slow rate part of the controller by taking into consideration closed-loop performance rather than merely approximating the controller in an open-loop sense. The two-stage approach can be thought as an automated approach to produce the customary notch filters used to reduce aliasing in a systematic and optimal manner within the $\mathcal{H}_\infty$ problem formulation.

The proposed approach and the resampling approach in which the low frequency part of the controller is downsampled before implementation are compared on two design examples. Our approach achieves better results both in terms of minimizing aliasing in the closed-loop system measured by computing the distance of the closed-loop multirate system design to the fast rate closed-loop system and in terms of closed-loop performance measured by comparing the time-domain closed-loop responses of the multirate designs to the performance of the fast rate system.
Chapter 4

Design of Causal Multirate $\mathcal{H}_\infty$ Optimal Controllers

The implementation of multirate controllers is constrained by the fact that current controller outputs cannot depend on future controller inputs. Such constraint is the so-called causality constraint, which needs to be imposed and considered through the controller design process. In [24] the causality constraint is specified in terms of all stabilizing controllers, thus only the feedthrough term of the state-space controller realization (i.e., D matrix) is constrained. A different approach [54] solves the problem by transforming the $\mathcal{H}_\infty$ design and characterizing the multirate $\mathcal{H}_\infty$ optimal controller in terms of two coupled Discrete Algebraic Ricatti equations with a set of associated matrix positive definiteness conditions. In such approach the causality condition is automatically satisfied.

The proposed design approach of causal $\mathcal{H}_\infty$ controllers is based on the Youla parametrization of all stabilizing controllers [6], [7] and the causality constraint derived in [24]. The $\mathcal{H}_\infty$ optimization problem is reformulated so that the obtained stabilizing controller minimizing the $\mathcal{H}_\infty$ closed loop norm also satisfies the causality condition (see for example [55] and references therein). The one block GDP with
causality constraints was solved also in [56]. The optimization problem is then solved using the Ellipsoid Algorithm (EA) [21], [22].

The remainder of this Chapter describes the steps for the design of causal multirate $\mathcal{H}_\infty$ optimal controllers and ends with a design example that validates the proposed approach. The parametrization of all stabilizing controllers is reviewed in Section 4.1 while Section 4.2 extends the known concept of causality in LTI systems to multirate systems thus ensuring proper implementation. Section 4.3 overviews the Ellipsoid Algorithm, which is used to solve the optimization problem obtained from the application of the causality constraint in the $\mathcal{H}_\infty$ optimization problem. The algorithm for the design of causal multirate $\mathcal{H}_\infty$ optimal controllers is presented in Section 4.4. Finally, Section 4.5 presents a design example that validates the methods and algorithms presented along this Chapter.

4.1 Youla parametrization review

Consider a discrete-time system $P(z)$ with exogenous inputs $w$ and controls $u$ and exogenous outputs $z$ and measurements $y$ as shown in Figure 4.1, all of them with the appropriate dimensions. Assume that $P(z)$ has minimal state-space representation as follows

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$ (4.1)

Figure 4.1 shows the basic feedback configuration of the system $P(z)$ with the controller controller $K(z)$ to be designed. We can compute the closed loop relating $w$ and $z$ using a Linear Fractional Transformation (LFT) (see for example, Chapter 9 in [14]) thus

$$T_{zw}(z) = \mathcal{F}_l(P, K) = P_{11}(z) + P_{12}(z)K(z)[I - K(z)P_{22}(z)]^{-1}P_{21}(z)$$ (4.2)
It is also well known [8] that general robust control problems can be obtained by formulating them as an optimization problem of finding the stabilizing controller that minimizes \( \|T_{zw}(z)\|_{\infty} \). Assume that there exists a controller \( K(z) \) such that the closed loop \( T_{zw}(z) \) is stable. By properly formulating the optimization problem [14], \( T_{zw}(z) \) would also achieve the required performance specifications. This Section focuses on the parametrization of all stabilizing controllers which is defined as the set \( \mathcal{K} \). Such parametrization was first introduced by Youla in [6] and [7] using the coprime factorization technique.

Assume that \((A, B_2)\) is stabilizable and \((C_2, A)\) is detectable. Then for each controller \( K(z) \in \mathcal{K} \) there exists a \( Q(z) \in \mathcal{H}_\infty \) such that

\[
T_{zw}(z) = T_{11}(z) - T_{12}(z)Q(z)T_{21}(z)
\]  

(4.3)

where

\[
T = \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix}
= \begin{bmatrix} A + B_2F & -B_2F & B_1 & B_2R_b^{-1} \\ 0 & A + HC_2 & B_1 + HD_{21} & 0 \\ C_1 + D_{12}F & -D_{12}F & D_{11} & D_{12}R_b^{-1} \\ 0 & R_c^{-1}C_2 & R_c^{-1}D_{21} & 0 \end{bmatrix}
\]

(4.4)

with \( Y \) and \( F \) computed so \( A_F := A + B_2F \) is stable and solve the equations

\[
Y = A_F^TYYA_F + C_F^TC_F
\]

(4.5)

\[
0 = B_2^TYA_F + D_{12}^TC_F
\]

(4.6)
where $C_F := C_1 + D_{12}F$ and (4.5)-(4.6) are derived by solving a Discrete Algebraic Ricatti Equation (see [57], [58] and references therein).

Let $R_b = (D_{12}^T D_{12} + B_2^T Y B_2)^{1/2}$ and $R_{b \perp} = (D_{12 \perp}^T D_{12 \perp} + B_2^T Y B_\perp)^{1/2}$ where $D_{12 \perp}$ and $B_\perp$ solve the system

$$
\begin{bmatrix}
D_{12}^T & B_2^T Y \\
C_F^T & A_F^T Y \\
\end{bmatrix}
\begin{bmatrix}
D_{12 \perp} \\
B_\perp \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
$$

(4.7)

with $\begin{bmatrix} D_{12 \perp} \\ B_\perp \end{bmatrix}$ full column rank and as many columns as the number of rows of $C_F$ minus the number of columns of $B_2$.

On the other hand, define $B_H := B_1 + HD_{21}$ with H computed so $A + HC_2$ is stable and X and H solve the following equations

$$
X = A_H X A_H^T + B_H B_H^T \\
0 = C_2 X A_H^T + D_{21} B_H^T
$$

(4.8) (4.9)

which are also obtained by solving a Discrete Algebraic Ricatti Equation ([57], [58]).

Define $R_c := D_{21}^T D_{21}^T + C_2 X C_2^T)^{1/2}$ and let $C_{\perp}$, $D_{21 \perp}$ solve the following system

$$
\begin{bmatrix}
C_{\perp} & D_{21 \perp} \\
\end{bmatrix}
\begin{bmatrix}
X A_H^T & X C_2^T \\
B_H^T & D_{21}^T \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
\end{bmatrix}
$$

(4.10)

with $\begin{bmatrix} C_{\perp} \\ D_{21 \perp} \end{bmatrix}$ full row rank and as many rows as the number of columns of $B_1$ minus the number of row of $C_2$. We have then $R_{c \perp} := D_{21 \perp} D_{21 \perp}^T + C_\perp X C_{\perp \perp}^T)^{1/2}$.

Assuming that $D_{12}$ and $D_{21}$ are full column and row rank respectively it can be derived [59] that under some assumptions $T_{12}$ and $T_{21}$ can be chosen to be inner and co-inner respectively. There also exist $T_{12 \perp}$ and $T_{21 \perp}$ such that $\begin{bmatrix} T_{12} & T_{12 \perp} \end{bmatrix}$ and $\begin{bmatrix} T_{21} \\ T_{21 \perp} \end{bmatrix}$ are unitary.
Since multiplication by unitary transfer functions do not change the \( \mathcal{H}_\infty \) norm, it follows that
\[
\| T_{11} - T_{12}(z)Q(z)T_{21}(z) \|_\infty = \left\| \begin{bmatrix} T_{12\perp} & T_{12} \end{bmatrix} \sim T_{11} \begin{bmatrix} T_{21\perp} \\ T_{21} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \right\|_\infty
\]
\[
= \left\| G^\sim - \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \right\|_\infty,
\]
where
\[
G^\sim = \begin{bmatrix} T_{12\perp} & T_{12} \end{bmatrix} \sim T_{11} \begin{bmatrix} T_{21\perp} \\ T_{21} \end{bmatrix} \sim
\] (4.11)

From (4.11), the \( \mathcal{H}_\infty \) optimal control problem can be now formulated as the approximation problem:

Given \( G(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} \in \mathcal{RH}_\infty \), and a number \( \gamma \), find all \( Q \in \mathcal{RH}_\infty \) that achieve
\[
\left\| \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & Q^\sim(z) \end{bmatrix} \right\|_\infty \leq \gamma
\] (4.13)

\( \gamma_0 \) is said to be optimal if it is the infimum number for which the problem in (4.13) has a feasible solution.

The optimal control problem presented above is the so-called General Distance Problem (GDP) [60]. If all \( G_{ij}(z) \)'s are nonzero, (4.13) is called a Four Block problem [61]. On the other hand, if \( G_{11}(z) \), \( G_{12}(z) \), and \( G_{21}(z) \) are all equal to zero then the optimal control problem is called One Block problem also known as the Nehari Problem (NP) [8]. The One Block problem can be solved and a parametrization of all the solutions can be computed while in the Four Block problem for a given (constant) \( \gamma \) we can find also a solution \( Q(z) \in \mathcal{RH}_\infty \) [58].

Consider that a solution \( Q(z) \in \mathcal{RH}_\infty \) is computed for the optimal problem
(4.13) and that has a state-space representation as

\[
Q(z) = \begin{bmatrix}
A_Q & B_Q \\
C_Q & D_Q
\end{bmatrix}
\]  

(4.14)

The desired controller \(K(z)\) with state-space representation

\[
K(z) = \begin{bmatrix}
A_K & B_K \\
C_K & D_K
\end{bmatrix}
\]  

(4.15)

can be obtained by computing the LFT of \(Q(z)\) with the system \(J(z)\) (i.e., \(K(z) = \mathcal{F}_l(J(z), Q(z))\)) where \(J(z)\) is computed from the system parameters used in the controller parametrization and has state-space representation as follows

\[
J(z) = \begin{bmatrix}
A_J & B_J \\
C_J & D_J
\end{bmatrix} = \begin{bmatrix}
A + B_2F + HC_2 + HD_{22}F & -H & B_2 + HD_{22}R_b^{-1} \\
F & 0 & R_b^{-1} \\
-R_c^{-1}(C_2 + D_2F) & R_c^{-1} & R_c^{-1}D_{22}R_b^{-1}
\end{bmatrix}
\]  

(4.16)

In the following section we present the causality constraints in multirate systems; thus we establish the conditions needed on \(Q(z)\) so the consequent multirate controller \(K(z)\) is a causal multirate \(H_\infty\) optimal controller.

### 4.2 Causality constraints in multirate systems

![Figure 4.2: Lifted representation of the system \(P_{ji}(z)\).](image)

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Consider a lifted multirate system $P_{ji}(z)$ as obtained in Chapter 2, such system is shown in Figure 4.2. It can be shown [24] that the expressions in Section 2.2 satisfy the following condition:

$$[D_{ji}]_{\alpha\beta} = 0 \quad \text{if} \quad (\alpha - 1)n_i - (\beta - 1)m_j < 0 \quad \text{with} \quad 1 \leq \alpha \leq \overline{n}_i, \quad 1 \leq \beta \leq \overline{m}_j \quad (4.17)$$

Condition (4.18) expresses the $(\overline{n}_i, \overline{m}_j)$ causality constraint in multirate systems.

In addition, it also yields that if $D_{ji}^{(1)}$ and $D_{ji}^{(2)}$ satisfy the $(\overline{n}_i^{(1)}, \overline{m}_j^{(1)})$ and $D_{ji}^{(2)}$ satisfies the $(\overline{n}_i^{(2)}, \overline{m}_j^{(2)})$ causality condition respectively, then $D_{ji}^{(1)}D_{ji}^{(2)}$ satisfies also the $(\overline{n}_i^{(1)}, \overline{m}_j^{(2)})$ causality constraint.

Define $Q_0 := Q(\infty) \equiv D_Q$ for $Q(z) \in \mathcal{RH}_\infty$ obtained after solving the general distance problem in (4.13). Assume also that $Q(z)$ has $\overline{m}_j$ inputs and $\overline{n}_i$ outputs and satisfies the $(\overline{m}_j, \overline{n}_i)$ causality constraint in (4.18) thus

$$[Q_0]_{\alpha\beta} = 0 \quad \text{if} \quad (\alpha - 1)n_i - (\beta - 1)m_j < 0 \quad \text{with} \quad 1 \leq \alpha \leq \overline{n}_i, \quad 1 \leq \beta \leq \overline{m}_j \quad (4.18)$$

Using the coprime factorization approach, it can be shown [24] that such $Q(z)$ system leads to a controller $K(z) = \mathcal{F}_i(J(z), Q(z))$, where $J(z)$ is as defined in (4.16), that also satisfies the causality condition (i.e., $D_K$ satisfies $(\overline{m}_j, \overline{n}_i)$).

In the design of causal multirate $\mathcal{H}_\infty$ optimal controllers we constraint the solution of the GDP to be causal. More precisely, we can express $Q(z)$ in terms of its Markov elements $Q_0, Q_1, Q_2, \ldots$ such that $Q(z) = \sum_{k=0}^{\infty} Q_k z^{-k}$. Since the causality constraint only affects at $Q_0 \equiv D_Q$ as shown above, the optimization problem can be now formulated as

$$\text{Given } G(z) \in \mathcal{RH}_\infty, \text{ and a number } \gamma, \text{ find } Q = \begin{bmatrix} A_Q & B_Q \\ C_Q & D_Q \end{bmatrix} \in \mathcal{RH}_\infty \text{ such that } D_Q \text{ satisfies the causality constraint in } (4.13) \text{ and }$$

$$\|G(z) - Q^*(z)\|_\infty \leq \gamma \quad (4.19)$$
It is shown in [16] that such constrained optimization problem is convex. Moreover, it involves an optimization problem with constraints over all \( Q_i \)'s. We use the Ellipsoid Algorithm (EA) to solve it. The implementation of the EA was done in Fortran \(^{®} [58]\) and later translated to Matlab \(^{®} [62]\) in order to solve time-domain constrained \( H_\infty \) optimization problems. The next Section reviews the Ellipsoid Algorithm used for the resolution of constrained optimization problems.

### 4.3 Ellipsoid algorithm overview

In order to solve the constrained nondifferentiable problem presented in (4.19) we use the Ellipsoid Algorithm. Assume that the objective is to find a \( n \)-vector \( x \) minimizing a function \( \phi(x) \) and satisfying the constraint

\[
A^T x \leq b
\]

where \( A^T \) is a \( m \times n \) matrix and \( b \) is an \( m \)-vector. Assume also that \( n > 1 \). We can rewrite (4.20) as

\[
\begin{bmatrix}
a_1^T \\
a_2^T \\
\vdots \\
a_m^T
\end{bmatrix} x \leq \begin{bmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_m
\end{bmatrix}
\]

(4.21)

where columns of \( A \) are denoted as \( a_1, a_2, \ldots, a_m \) and the components of \( b \) are represented as \( \beta_1, \beta_2, \ldots, \beta_m \). Each of the \( a_i \) corresponds to outwards normals to the constraints. Furthermore, we can also rewrite (4.21) as

\[
a_i^T x \leq \beta_i \quad i = 1, 2, \ldots, m
\]

(4.22)

An ellipsoid \( E_k \) centered at the point \( x_k \) can be described as

\[
E_k = \{ x \in \mathbb{R}^n \mid (x - x_k)^T B_k^{-1}(x - x_k) \leq 1 \}
\]

(4.23)
where $B_k$ is a positive definite symmetric matrix. Notice that the square roots of
the eigenvalues of $B_k$ are the lengths of the semi-axes of the ellipsoid $E_k$.

The Ellipsoid Algorithm calculates a sequence of ellipsoids ($E_0$, $E_1$, \ldots) with
decreasing volume. Each of these $E_k$ ellipsoids will contain the optimal solution for
our problem if the first $E_0$ ellipsoid contains it.

At each $(kth)$ iteration the algorithm checks the center $x_k$ of the corresponding
ellipsoid for being a feasible solution (i.e., satisfies all the constraints). If it is not
feasible, one of the violated constraints is selected and a new ellipsoid is calculated.
More precisely, assume that some constraint violated by $x_k$ is
\begin{equation}
    a^T x \leq \beta
\end{equation}

Such constraint is used to compute following ellipsoid:
\begin{equation}
    \{ x \in E_k \mid a^T x \leq a^T x_k \}
\end{equation}

This new ellipsoid (4.25) is the ellipsoid of minimum volume that contains the half
ellipsoid defined by the violated constraint (4.24). The minimum volume ellipsoid
becomes the ellipsoid used in the next ($(k+1)th$) iteration. Therefore we can express
the elements of the new ellipsoid as follows:
\begin{align}
x_{k+1} &= x_k - \tau \left( \frac{B_k a}{\sqrt{a^T B_k a}} \right) \\
B_{k+1} &= \delta \left( B_k - \sigma \left( \frac{B_k a a^T B_k}{a^T B_k a} \right) \right)
\end{align}

where
\begin{align}
\tau &= \frac{1}{n+1} \\
\sigma &= \frac{2}{n+1} \\
\delta &= \frac{n^2}{n^2 - 1}
\end{align}

are the so-called step, dilation, and expansion parameters respectively.
If the point is feasible, the objective function \( \phi(x_k) \) and its generalized gradient \( g_k \in \partial \phi(x_k) \) are computed. Then, using the generalized gradient, the half ellipsoid is selected ensuring that contains the optimal solution of the problem. We compute the minimum volume ellipsoid containing the half ellipsoid previously selected as

\[
x_{k+1} = x_k - \tau \left( \frac{g_k}{\sqrt{g_k^TB_kg_k}} \right)
\]

\[
B_{k+1} = \delta \left( B_k - \sigma \left( \frac{B_kg_kg_k^TB_k}{g_k^TB_kg_k} \right) \right)
\]

where \( \tau, \sigma, \) and \( \delta \) are as defined in (4.28) - (4.30). Again, this new minimum volume ellipsoid becomes the ellipsoid used in the next iteration.

Modified versions of the ellipsoid algorithm use the so-called *Deep-cuts*, that are characterized by using less than half of the ellipsoid (but still containing the optimal solution). More specifically, assume that \( x_k \) violates the constraint in (4.22). The *deep cut* used is then \( a^T x \leq \beta \) instead of the *cut* \( a^T x \leq a^T x \) used in the basic algorithm, which passes through the center of \( E_k \). The obtained smallest ellipsoid \( E_{k+1} \) is given by \( x_{k+1} \) and \( B_{k+1} \), which are computed as in (4.26) and (4.27) but the parameters \( \tau, \sigma, \) and \( \delta \) are defined as

\[
\tau = \frac{1 + n\alpha}{n + 1}
\]

\[
\sigma = \frac{2(1 + n\alpha)}{(n + 1)(1 + \alpha)}
\]

\[
\delta = \frac{n^2(1 - \alpha^2)}{n^2 - 1}
\]

where

\[
\alpha = \frac{a^T x_k - \beta}{\sqrt{a^T B_k a}}
\]

The *Deep-cuts* modified algorithm can speed up the convergence rate of the algorithm. The interested reader can find more detailed information about the Ellipsoid algorithm and a proof of convergence in [21], and [22].

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4.4 Causal multirate $\mathcal{H}_\infty$ optimal controller design algorithm

In the previous sections of this Chapter we have presented the tools and conditions needed for the design of causal multirate $\mathcal{H}_\infty$ optimal controllers. This section focuses in the algorithm used for the computation of such controllers.

Consider a lifted multirate system $P(z)$ that has been computed as explained in Chapter 2. Such system has $p$ and $q$ blocked exogenous inputs and outputs respectively. Each input $i$ has dimension $\pi_i$ while each output $j$ has dimension $\nu_j$. System $P(z)$ has also a control signal $u$ and a measurement signal $y$ of dimension $\nu_u$ and $\nu_y$ respectively. Therefore $P(z)$ has a total number of inputs of $\nu_u + \sum_{i=1}^{p} \nu_i$ while the total number of outputs is $\nu_y + \sum_{j=1}^{q} \nu_j$. The measurement and control signals are used for the feedback interconnection with a controller $K(z)$ as shown in Figure 4.3.

Assume that the system $P(z)$ has been computed from an appropriated formulated $\mathcal{H}_\infty$ optimization problem such that some performance specifications need to be satisfied. Moreover, we want to design an $\mathcal{H}_\infty$ optimal controller that solves the
problem
\[
\min_{K(z) \text{ stabilizing}} \| F(P(z), K(z)) \|_\infty
\]

Notice that by the nature of the problem the controller \( K(z) \) is going to be a multirate controller. In addition, to ensure proper implementation the controller needs to be causal so its current output does not depend on future inputs. The causal multirate \( \mathcal{H}_\infty \) optimal controller proposed in this Chapter is obtained via the following steps:

1. **Unconstrained Design:** A first \( \mathcal{H}_\infty \) optimal controller design is performed. Such design is done by parametrizing all the stabilizing controllers (i.e., finding \( Q(z) \)). Such controller achieves a value \( \gamma_0 \) in (4.13). If the resulting system \( Q(z) \) has a \( Q_0 \) that satisfies the causality condition, the algorithm computes the controller \( K(z) \) and terminates execution here. Otherwise it continues to the following steps.

2. **Computation of the initial candidate solution for the Ellipsoid Algorithm:** The non causal elements of the matrix \( Q_0 \) corresponding to the unconstrained solution \( Q(z) \) are substituted with zeros obtaining \( Q_{c0} \), thus it now satisfies the causality condition. \( Q_{c0} \) is used as initial guess for the Ellipsoid algorithm.

3. **Optimization via Ellipsoid Algorithm:** Using the Ellipsoid Algorithm the constrained problem is solved. The problem is solved by minimizing the \( \mathcal{H}_\infty \) norm of the closed loop by considering as variables only the causal elements of the corresponding \( Q_0^{(i)} \) of the \( i \)th iteration. Once the Ellipsoid algorithm finds a solution it generates the needed system \( Q(z) \) with causal \( Q_0 \).

4. **Computation of the multirate controller, closed loop and \( \gamma \) achieved:** Using the obtained system \( Q(z) \) and the system \( J(z) \) computed in Step 1 the controller \( K(z) \) is obtained, as well as the closed loop \( F(P(z), K(z)) \) and its \( \mathcal{H}_\infty \) norm.
### 4.5 SISO Causal Multirate Design Example

The algorithm explained in the previous sections is validated through a design example. The fast rate controller is a SISO controller and the plant and formulation is the same as used in Section 3.3.1. As a reminder, the open-loop plant has four modes and the design objective is to robustly reject impulse disturbances at the measurement. The fast sampling rate \((1/T_f)\) used is 50kHz and we have a multirate ratio of \(N = 3\) (i.e. the slow sampling rate is \(1/3T_f = 1/T_s\)). The plant transfer function (discretized at fast sampling rate) is given in (4.37).

\[
P(z) = -2.0178 \times 10^{-5} \cdot \frac{(z + 16.87)(z + 2.336)(z + 0.805)(z + 0.376)}{(z^2 - 1.982z + 0.982)(z^2 - 1.471z + 0.978)} \cdot \frac{(z + 0.060)(z^2 - 0.779z + 0.946)}{(z^2 - 1.006z + 0.963)(z^2 - 0.237z + 0.865)}
\]  

(4.37)

Five controllers (one fast rate and four multirate) are designed. The multirate controllers are designed following the proposed algorithm to achieve causal multirate controllers but considering whether the measurement and the control signals are sampled at the fast or the slow sampling rate. Such considerations lead to the following notation of the designed controllers:

- **\(K_{FR}(z)\):** Fast rate \((1/T_f)\) controller designed of the fast rate plant \(P(z)\)
- **\(K_{Full}^{MR}(z)\):** Causal multirate controller designed to compensate the plant \(P(z)\).

  In this case, both the measurement and the control are considered fast rate signals but the system is lifted before the design thus obtaining a multirate controller.

- **\(K_{Meas}^{MR}(z)\):** Causal multirate controller designed to compensate the plant \(P(z)\) by considering the measurement as a slow rate \((1/T_s)\) signal.

- **\(K_{Cont}^{MR}(z)\):** Causal multirate controller designed to compensate the plant \(P(z)\) by considering the control as a slow rate \((1/T_s)\) signal.
Table 4.1: $\mu_{\infty}^{CL}$ for the causal multirate $H_\infty$ optimal controllers

<table>
<thead>
<tr>
<th></th>
<th>$\mu_{\infty}^{CL}$</th>
<th>$\mu_{\infty} \geq$</th>
<th>$\mu_{\infty} \leq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CL_{MR}^{Full}(z)$</td>
<td>9.1693 $\cdot$ 10^{-5}</td>
<td>3.1402 $\cdot$ 10^{-5}</td>
<td>3.2056 $\cdot$ 10^{-5}</td>
</tr>
<tr>
<td>$CL_{MR}^{Meas}(z)$</td>
<td>1.6304</td>
<td>1.6282</td>
<td>1.6282</td>
</tr>
<tr>
<td>$CL_{MR}^{Cont}(z)$</td>
<td>0.0813</td>
<td>0.0393</td>
<td>0.0393</td>
</tr>
<tr>
<td>$CL_{MR}^{Meas/Cont}(z)$</td>
<td>1.6381</td>
<td>1.6358</td>
<td>1.6358</td>
</tr>
</tbody>
</table>

- $K_{MR}^{Meas/Cont}(z)$: LTI slow rate controller designed to compensate the plant $P(z)$ by considering both the control and the measurement slow rate $(1/T_s)$ signals.
  
  We remark here that, although the controller is slow rate, the problem is formulated using a multirate plant since signals $w$ and $z$ are fast rate signals.

These controllers lead to the closed loops $CL_{FR}(z)$, $CL_{MR}^{Full}(z)$, $CL_{MR}^{Meas}(z)$, $CL_{MR}^{Cont}(z)$, and $CL_{MR}^{Meas/Cont}(z)$ respectively.

We evaluate the performance of the controllers using the (2.82) thus measuring the distance of the corresponding closed loop to the fast rate closed loop as

$$\mu_{\infty}^{CL} := \|CL_{FR} - CL_{MR}^{j}\|_{\infty} \quad j = \text{Full, Cont, Meas, Meas/Cont}. \quad (4.38)$$

Since the multirate ratio $N$ is three, we also compute the bounds on $\mu_{\infty}$ as in (2.83) and (2.84). Table 4.1 summarizes the results for the four multirate controllers.

We can see in Table 4.1 that the Full causal multirate controller $K_{MR}^{Full}(z)$ leads to a closed loop very close to the fast rate controller $K_{FR}(z)$ and the bounds are also very small, which allow us to affirm that the aliasing content is very small also. From the other three controllers, we can see that the distance is considerably smaller for the controller in which the controls are downsampled to slow rate $(1/T_s)$. On the other hand, it seems that downsampling the measurements has a major effect on the performance of the controller, both in distance of the closed loop and aliasing bounds. Such effect overtakes the downsampling of the control.
We also compare the time responses of the closed loop plant while applying an impulse disturbance in the measurement. We can see in Figure 4.4 that the simulations confirm the results analyzed previously from Table 4.1. The Full causal multirate controller $K_{MR}^{Full}(z)$ leads to a disturbance rejection performance equal to the one from the fast rate controller $K_{FR}(z)$. We can also see that the controller $K_{MR}^{Cont}(z)$ compensates for the disturbance with similar performance as the fast rate controller $K_{FR}(z)$. On the other hand, both controllers for which the measurement signal was considered a slow rate signal ($CL_{Meas}^{MR}(z)$ and $CL_{Meas/Cont}^{MR}(z)$) have the worst performance being also very similar to each other.

![Figure 4.4: Time response for the SISO causal multirate $\mathcal{H}_\infty$ optimal controllers](image)

Finally, we analyze the complexity of the controllers obtained. We have found that we can effectively model reduce the controller order by truncating stable modes while assuring a maximum $\mathcal{H}_\infty$ norm of the error between the initial and the order reduced system to be $10^{-6}$. Assume that the computation load a system is proportional to number of multiplications per sample period done for the update and computation of its inputs, outputs, and states ($n_i, n_o$, and $n_x$ respectively). Further...
thermore, we can approximate the computational load associated to the input and output update as

\[ C_{io} \approx n_i \frac{N}{m_i} m_o \frac{N}{m_o} + n_x n_i \frac{N}{m_i} + n_x n_o \frac{N}{m_o} \]  

(4.39)

where \( T_i := m_i T_f \), and \( T_o = m_o T_f \) are the period at which the inputs and outputs and states are updated in the system and \( N \) is the multirate ratio (i.e., ratio between the slow and fast sampling periods \( N = T_s / T_f \). Assuming that we have a state-space realization in the companion form, we can approximate the number of multiplications associated with the number of states as \( C_s \approx n_x \).

We define the computational load associated to a system as \( C_l := C_{io}^l + C_s^l \) where \( l \) denotes the type of system studied; either fast rate (FR), Multirate (MR) or slow rate (SR). Thus we have

\[
C_l := n_i \frac{N}{m_i} m_o \frac{N}{m_o} + n_x n_i \frac{N}{m_i} + n_x n_o \frac{N}{m_o} + n_x \\
= N^2 \frac{n_i n_o}{m_i m_o} + n_x \left( \frac{N n_i}{m_i} + \frac{N n_o}{m_o} + 1 \right)
\]

For the controllers that have been designed, we obtain the following computational loads

\[
C_{FR} = 1 + 3n_x^{FR} \quad (4.40)
\]
\[
C_{Full}^{MR} = 9 + 7n_x^{Full} \quad (4.41)
\]
\[
C_{Meas}^{MR} = 3 + 5n_x^{Meas} \quad (4.42)
\]
\[
C_{Cont}^{MR} = 3 + 5n_x^{Cont} \quad (4.43)
\]
\[
C_{Meas/Cont}^{MR} = 1 + 3n_x^{Meas/Cont} \quad (4.44)
\]

where the multirate ratio used has been \( N = 3 \). Notice that we also need to consider that the number multiplications done in the fast sampling period \( T_f \). Thus the computational load \( C_{FR} \) is multiplied by \( N = 3 \) so it is comparable to the computational load of the multirate controllers, which are implemented at slow
sampling period $T_s$. We define the computational load percentage as follows:

$$\tilde{C}_{MR}^j := 100 \times \frac{C_{MR}^j}{N C_{FR}} \% , \; j = \text{Full, Cont, Meas, Meas/Cont}. \quad (4.45)$$

The computational load percentage $\tilde{C}_{FR}$ in (4.45) measures the percentage of computational load that a multirate controller has with respect of the computational load of a fast rate controller. Table 4.2 shows the number of modes and computational loads for each of the designed controllers. Notice that the computational load $\tilde{C}_{FR}$ of the fast rate controller $K_{FR}(z)$ is 100%. In addition, the fast rate $\mathcal{H}_\infty$ optimal controller $K_{FR}(z)$ obtained has thirteen modes but can be effectively model reduced to ten. The results in Table 4.2 show that the computational load can be reduced to a 92% of the fast rate computational load of the fast rate controller $K_{FR}(z)$ using the Full multirate controller $K_{MR}^{\text{Full}}(z)$ and to a 36% using the slow rate controller $K_{MR}^{\text{Meas/Cont}}(z)$.

Finally, we evaluate the computational load percentage (4.45) and the performance loss (3.31) of the causal multirate controllers for different values of the multirate ratio $N$. The results are shown in 4.5. We can see that the Full multirate controller $K_{MR}^{\text{Full}}(z)$ does not have any performance loss and a computational load reduction around 90%. Notice also that the controller $K_{MR}^{\text{Cont}}(z)$ has a performance loss that does not go above 0.25% but achieve a computational load reduction of 50% for a multirate ratio $N = 8$. 

<table>
<thead>
<tr>
<th>$n_{MR}^{\text{des}}$</th>
<th>$n_{MR}^{\text{red}}$</th>
<th>$\tilde{C}_{MR}^j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_{MR}^{\text{Full}}(z)$</td>
<td>13</td>
<td>11</td>
</tr>
<tr>
<td>$K_{MR}^{\text{Meas}}(z)$</td>
<td>15</td>
<td>12</td>
</tr>
<tr>
<td>$K_{MR}^{\text{Cont}}(z)$</td>
<td>14</td>
<td>12</td>
</tr>
<tr>
<td>$K_{MR}^{\text{Meas/Cont}}(z)$</td>
<td>13</td>
<td>11</td>
</tr>
</tbody>
</table>

$n_{MR}^{\text{des}}$ and $n_{MR}^{\text{red}}$ are the number of modes obtained from the design and after model reduction respectively.
Figure 4.5: Performance loss and computational load reduction of the multirate controllers for different values of the multirate ratio $N$
Chapter 5

Lifting Based Approach in the Design of Multirate Filter Banks

A digital filter bank is formed by $M$ digital filters with either common input or output. The former configuration is known as Analysis bank and its usually denoted as $H_l(z)$ while the latter is so called the Synthesis bank and denoted by $F_l(z)$ with $l = 1, \ldots, M - 1$ [2]. Furthermore, each of the $H_l(z)$ and $F_l(z)$ filters are known as the analysis and synthesis filters respectively.

The analysis filter bank splits a signal $x(k)$ into $M$ signals $x_l(k)$ which are commonly called subband signals. On the other hand, the synthesis filter bank combines the $M$ subbands signals $x_l(k)$ into a single signal $\hat{x}(k)$. Figure 5.1 shows both the analysis and synthesis filters.

Typically, each synthesis and analysis filter is a bandpass filter, thus the filter bank is intended to avoid overlapping between frequency bands in its output signals, such effect is also known as aliasing. Indeed, the application for which a filter bank is designed determines the amount of overlapping between frequency bands that can be tolerated. Speech coding, signal transmission, or adaptive signal processing are
some of the disciplines where filter banks are used.

Analysis and synthesis filter banks are used together in the so-called maximally
decimated filter banks [63]. The input signal \( x(k) \) is filtered by the analysis filter
\( H_l(z) \) bank, creating the subband signal \( x_l(k) \). Each signal is downsampled (i.e.,
decimated) to produce the signal \( v_l(k) \). The signal \( v_l(k) \) obtained is such that has
limited frequency content thus can be easily coded and transmitted. The signal
is interpreted by receiver and then upsampled creating the signal creating
\( y_l(k) \).

Signal \( y_l(k) \) in filtered by the synthesis filter \( F_l(z) \) bank creating the resulting signal
\( \hat{x}(k) \).

The idea of such system was first introduced in [41] and is also known as a \( M \)
channel filter bank. Figure 5.2 shows the representation of a \( M \) channel filter bank.
Although the coding and transmission processes are not shown, they occur between
the downsampled and upsampled blocks.

In this Chapter, we focus in the obtention of both analysis and synthesis filter
banks so the error (i.e., distortion [43]) between \( x(k) \) and \( \hat{x}(k) \) is minimized. Such
property leads to the so called Perfect Reconstruction (PR) filter banks [42]. Such
systems have been studied [42], [43] from the digital signal processing point of view.
Such approach is based on the polyphase representation [2].

The design of filter banks as an optimization problem was presented in [44],
where a model matching problem was solved using the $\mathcal{H}_\infty$ norm as the performance measure. An $m$ delay $(z^{-m})$ was the reference model to match, since it can exemplify a transmission delay of a signal between two points. An important fact is also that the $M$ channel filter bank is identified as a multirate filter bank, since the problem involve two different sampling periods. Moreover, nonuniform multirate filter banks are designed in [45]. In a nonuniform filter bank the sampling rates used are different within the analysis and the synthesis filter bank. In both cases, the design is based on the use of the polyphase matrices and delay chains as shown in Figure 5.3.

The systems $E(z)$ and $R(z)$ are related to the analysis and synthesis filter banks.
as

\[
\begin{bmatrix}
H_0(z) \\
H_1(z) \\
\vdots \\
H_{M-1}(z)
\end{bmatrix}
= \begin{bmatrix}
E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0,M-1}(z^M) \\
E_{10}(z^M) & E_{11}(z^M) & \cdots & E_{1,M-1}(z^M) \\
\vdots & \vdots & \ddots & \vdots \\
E_{M-1,0}(z^M) & E_{M-1,1}(z^M) & \cdots & E_{M-1,M-1}(z^M)
\end{bmatrix}
\begin{bmatrix}
1 \\
z^{-1} \\
\vdots \\
z^{-(M-1)}
\end{bmatrix}
\] (5.1)

and

\[
\begin{bmatrix}
F_0(z) & F_1(z) & \cdots & F_{M-1}(z)
\end{bmatrix}
= \begin{bmatrix}
1 & z^{-1} & \cdots & z^{-(M-1)}
\end{bmatrix}.
\] (5.2)

Notice that \(H(z)\) is a 1 input \(M\) outputs system while \(F(z)\) is a system that has \(M\) inputs and 1 output.

The delay chains in Figure 5.3 cause a delay in the signal before downsampling thus, in practice, the output depends on past signals. That gives the advantage of not having the causality constraints when designing the filter but it has the disadvantage of having less updated input signal information to compute the output. Moreover, the performance could be worse when not comparing to a pure \(m\) delay \(z^{-m}\). Another fundamental difference of the digital signal approach from the proposed approach is that the upsampled signal is not held (i.e., repeated) during the fastest sampling period. That gives rise to fast rate signals with zeros in them.

The proposed design approach considers each filter bank \((H(z)\text{ and } F(z))\) as a multirate system and as such, we can compute its lifted representation (as derived
in Section 2.2). Figure 5.4 shows the transformations needed to compute the lifted filter bank systems $H(z)$ and $F(z)$.

The lifted analysis filter bank $H(z)$ and the lifted synthesis filter bank $F(z)$ will be used in the following sections to present the filter bank design approach. The model to match is a $m$ delay $z^{-m}$. The problem is formulated as a general distance problem using lifted systems. In addition, the design can be easily automated and an iterative design procedure is also presented. The design starts with a known analysis filter bank thus the synthesis filter bank is designed. The first iteration keeps the designed synthesis filter bank and redesigns the analysis filter bank if the $H_{\infty}$ performance measure can be improved. Furthermore, since the design is formulated using lifted systems, the designed filter is constrained to be causal. Such constraint is applied by using the ellipsoid algorithm presented in Section 4.3.

The remainder of this Chapter is organized as follows. Section 5.1 presents the proposed approach for the design of filter banks as a model matching $H_{\infty}$ optimization problem. First in Section 5.1.1, the approach for the design of the synthesis filter bank is derived followed by the analysis filter banks in Section 5.1.2. Section 5.1.3 presents the iterative approach for the design on multirate filter banks. Three different design examples using the proposed approach are shown in Section 5.2. Finally,
conclusions are given in Section 5.3.

## 5.1 Proposed Design Approach

Define $T_{dm}(z) := z^{-m}$ as the transfer function of a $m$ delay. The design objective is to obtain an analysis filter bank and synthesis filter bank for some integer $m \geq 0$ such that the difference between the desired output signal $x(k-m)$ and the actual output signal $\hat{x}(k)$ is as small as possible. It can be said then that perfect reconstruction has been achieved. We also define the error signal as $e(k) := x(k-m) - \hat{x}(k)$. Such configuration is depicted in Figure 5.5.

![Figure 5.5: Design approach using a $M$ channel filter bank.](image)

Using the lifting technique presented in Chapter 2, we transform the systems in Figure 5.5 to their lifted version. We obtain then the lifted delay, analysis and synthesis lifted filter banks represented in Figure 5.6 as $T_{dm}(z)$, $H(z)$, and $F(z)$ respectively.

The proposed design approach is formulated as a $\mathcal{H}_\infty$ optimization problem thus we want to obtain either the analysis filter bank $H(z)$ or the synthesis filter bank $F(z)$ that minimizes the $\mathcal{H}_\infty$ norm of the transfer function $M(z)$ relating the input
Figure 5.6: Design approach using the lifted systems.

signal $x(k)$ and the error signal $e(k)$. We can compute it as follows:

$$
\|M(z)\|_{\infty} = \|T_{dm}(z) - F(z)H(z)\|_{\infty} = \|T_{dm}(z) - F(z)H(z)\|_{\infty} \quad (5.3)
$$

In the following two sections we reformulate the $\mathcal{H}_\infty$ optimization model matching problem to a one block General Distance Problem. The sections are based on whether we want to compute the analysis or the synthesis filter bank. We remark that problem needs to be solved so the lifted filter bank obtained is causal and also stable.

5.1.1 Synthesis Filters Design

Assume that the lifted system $H(z)$ can be written as $H(z) = H_o(z)H_i(z)$ where is $H_o(z)$ c0-outer and $H_i(z)$ co-inner [8]. We remark that a system $G(z) \in \mathcal{RH}_\infty$ is said to be co-inner if $G(z)G(z)^\sim = I$. Since multiplication by unitary systems does not change the $\mathcal{H}_\infty$ norm, we have that

$$
\|M(z)\|_{\infty} = \|T_{dm}(z) - F(z)H(z)\|_{\infty} = \|T_{dm}(z) - F(z)\tilde{H}(z)\|_{\infty}
$$

$$
= \|T_{dm}(z) - F(z)H_o(z)H_i(z)\|_{\infty}
$$

$$
= \|T_{dm}(z)\left[ \begin{array}{c} H_o^\sim(z) \\ H_i(z) \end{array} \right] - F(z)H_o(z)H_i(z)\left[ \begin{array}{c} H_o^\sim(z) \\ H_i^\sim(z) \end{array} \right] \|_{\infty}
$$
\[
\begin{align*}
    &\|T_{dm}(z) \begin{bmatrix} H_t^\sim(z) & H_{a1}^\sim(z) \end{bmatrix} - \begin{bmatrix} F(z)H_0(z) & 0 \end{bmatrix} \|_\infty \\
    &= \|G_1(z) - \begin{bmatrix} F(z)H_0(z) & 0 \end{bmatrix} \|_\infty
\end{align*}
\]

where we define \( G_1(z) := T_{dm}(z) \begin{bmatrix} H_t^\sim(z) & H_{a1}^\sim(z) \end{bmatrix} \).

Assume that each analysis filter \( H_l(z) \) is a SISO filter and that we have a uniform \( M \) channel filter-bank. Moreover, the analysis filter-bank \( H(z) \) has one fast rate input and \( M \) slow rate outputs. Thus the lifted analysis filter-bank \( H(z) \) will have the same (i.e., \( M \)) number of inputs and outputs. Therefore (5.4) can be defined as

\[
\|G_1(z) - F(z)H_0(z)\|_\infty
\]

where \( G_1(z) := T_{dm}(z)H_t^\sim(z) \). Consider that such system \( G_1(z) \) can be decomposed into its stable and anti-stable components \( G_{1s}(z) \) and \( G_{1a}(z) \) respectively. Notice then that \( G_1(z) = G_{1a}(z) + G_{1s}(z) \). Therefore (5.5) can be rewritten as

\[
\|M(z)\|_\infty = \|G_1(z) - F(z)H_0(z)\|_\infty
\]

\[
= \|G_{1a}(z) + G_{1s}(z) - F(z)H_0(z)\|_\infty
\]

\[
= \|R_1(z) - Q_1^\sim(z)\|_\infty
\]

where

\[
R_1(z) = G_{1a}(z) \quad (5.7)
\]

\[
Q_1(z) = G_{1s}(z) - F(z)H_0(z) \quad (5.8)
\]

Notice that (5.6) is a one block General Distance Problem as presented in (4.13). Once the problem is solved, we can obtain the lifted synthesis filter bank as \( F(z) \) as

\[
F(z) = [G_{1s}(z) - Q_1(z)]H_t^{-1}(z)
\]

Once the lifted system \( F(z) \) is obtained, we can use the representations presented in Chapter 2 to obtain the synthesis filters \( F_l(z) \).
5.1.2 Analysis Filters Design

Similarly to the previous section, we can find the analysis filter $H(z)$ that minimizes the $\mathcal{H}_\infty$ in (5.3). Assume that the lifted system $E(z)$ can be now written as $E(z) = E_i(z)E_o(z)$ where is $E_o(z)$ outer and $E_i(z)$ inner [8]. Following the same approach as in (5.4) we have that

$$\|M(z)\|_\infty = \|T_{dm}(z) - F(z)H(z)\|_\infty$$

$$= T_{dm}(z) - F(z)E_o(z)H(z)$$

$$= \|[E_i(z)\quad E_i(z)\perp] T_{dm}(z) - [E_i(z)E_o(z)H(z)]\|_\infty$$

$$= \|[E_i(z)\quad E_i(z)\perp] T_{dm}(z) - [E_o(z)H(z)\quad 0]\|_\infty$$

$$= \|G_f(z) - [E_o(z)H(z)\quad 0]\|_\infty$$

(5.10)

where we define $G_f(z) := [E_i(z)\quad E_i(z)\perp] T_{dm}(z)$. Assume that we have SISO synthesis filters $F_i$ and an uniform $M$ channel filter-bank thus the lifted synthesis filter-bank $E(z)$ is a square system. Therefore, we can rewrite (5.10) as

$$\|M(z)\|_\infty = \|G_2(z) - E_o(z)H(z)\|_\infty$$

(5.11)

where $G_2(z) := E_i(z)T_{dm}(z)$.

Consider also that the system $G_2(z)$ can be decomposed into its stable and anti-stable components $G_{2s}(z)$ and $G_{2a}(z)$ respectively thus $G_2(z) = G_{2a}(z) + G_{2s}(z)$. Therefore (5.11) can be rewritten as

$$\|M(z)\|_\infty = \|G_2(z) - E_o(z)H(z)\|_\infty$$

(5.12)
where

\[ R_2(z) = \mathcal{G}_2(z) \]  
\[ Q_2(z) = \mathcal{G}_2(z) - \mathcal{F}_a(z)\mathcal{H}(z) \]  
\[ \text{(5.13)} \]

\[ \text{Notice that (5.6) is a general distance problem as presented in (4.13). Once the problem is solved, we can obtain the lifted synthesis filter bank as } \mathcal{F}(z) \text{ as} \]

\[ \mathcal{H}(z) = \mathcal{F}_a^{-1}(z)[\mathcal{G}_2(z) - Q_2(z)] \]  
\[ \text{(5.15)} \]

Once the lifted system \( \mathcal{H}(z) \) is obtained, we can use the representations derived in Chapter 2 to obtain the synthesis filters \( H_i(z) \).

### 5.1.3 Iterative filter bank design

As shown in the previous sections, we can design independently both the analysis and synthesis multirate filter banks. This section presents a simple iteration process that leads to the design of both analysis and synthesis multirate filter banks. The iteration process can be summarized in the following two steps:

1. **First synthesis filter bank design:** Assuming that a analysis filter bank \( H^{(0)}(z) \) is given. The synthesis filter bank \( F^{(0)}(z) \) is designed as derived in Section 5.1.1.

2. **Analysis and synthesis filter bank design:** The first iteration uses the synthesis filter bank designed in the previous step (i.e., \( F^{(0)}(z) \)). Each iteration is intended to obtain filter banks such that the \( H_\infty \) norm of the closed loop decreases the closed loop \( H_\infty \) norm from the previous iteration. Each iteration \( i \) is divided in two parts as follows

   a) The first part at each iteration designs the analysis filters \( H^{(i)}(z) \) using the
synthesis filters obtained in the previous iteration (i.e., \( F^{(i-1)}(z) \)). Such design is achieved as derived in Section 5.1.2.

b) The second part of the iteration \( i \) designs the synthesis filters \( F^{(i)}(z) \) using the analysis filters obtained in the first part of the same iteration \( i \) (i.e., \( H^{(i)}(z) \)).

The following section presents three examples using the design approach presented in this section. The two first examples assume that the analysis filters are designed previously and the synthesis filters are the ones to be designed. The third example uses both the analysis and synthesis design approach to iterate and obtain the multirate filter bank that locally minimizes the \( H_{\infty} \) norm of the error signal.

5.2 Design Examples

The validation of the design approach presented above is done with three filter banks design examples. For the designs, we consider a two channel filter bank as depicted in Figure 5.7. Typically, filters \( H_0(z) \) and \( H_1(z) \) are low and high pass filter respectively. A special filter bank configuration where the filters are designed so \( H_1(z) = H_0(-z) \), \( F_0(z) = H_0(z) \) and \( F_1(z) = -H_1(z) \) is known as a Quadrature Mirror Filter (QMF) bank [2].

The multirate filter banks are designed by the design approach proposed in Section 5.1. The first example defines the filter \( H_0(z) \) as a third order Butterworth filter.
lowpass filter with cutoff frequency $\pi/2$ while in the second example $H_0(z)$ is a Finite Impulse Response (FIR) linear phase filter of order 19. In both examples, the $H_1(z)$ filter is taken to be $H_1(z) = H_0(-z)$ and the synthesis filter banks are designed. In the third example, $H_0(z)$ and $H_1(z)$ are taken to be random third order systems. Using the iteration approach presented in Section 5.1.3 both analysis and synthesis filter banks are designed.

In addition, time domain simulations are presented for all the examples. We feed the two channel multirate filter bank with a signal $x(k)$ and measure the error signal $e(k)$. The input signal $x(k)$ is obtained by adding a low and a high frequency sinusoid so it yields

$$x(k) = \cos(0.5k) + \cos(2.5k) \quad (5.16)$$

Signal $x(k)$ is shown in Figure 5.8.

![Figure 5.8: $x(k)$ input signal.](image)

Using $x(k)$ as the input signal, the error signal $e(k)$ is obtained and plotted for the designed filter bank. On the other hand, its Root Mean Square (rms) is also computed.
5.2.1 Synthesis Filters Design Example 1

The first example uses a third order Butterworth lowpass filter with cutoff frequency $\pi/2$ as $H_0(z)$ and $H_1(z) = H_0(-z)$ thus obtaining a third order high pass filter. Figure 5.9 shows the magnitude bode plots versus normalized frequency for such $H_0(z)$ and $H_1(z)$ analysis filters.

![Figure 5.9: $|H_0(z)|$ and $|H_1(z)|$ filters versus normalized frequency for the design example 1.](image)

For the design example 1 we define $m = 9$ as done in [44]. The optimal norm achieved when designing the synthesis filter bank is $\|M(z)\|_\infty = 0.0124$, thus the relative rms error between the reconstructed signal $\hat{x}(k)$ and the desired signal $x(k-9)$ is less 1.3%. Figure 5.10 shows the magnitude bode plot of the synthesis filters $F_0(z)$ and $F_1(z)$. Notice that the obtained filters have a low pass ($F_0(z)$) and high pass ($F_1(z)$) behavior, which is to be expected from the initial analysis filter bank used in the design. The synthesis filters designed have greater order than the analysis filters used, being both $F_0(z)$ and $F_1(z)$ of eleventh order.

We also evaluate the performance of the filter bank by simulating a signal $x(k)$
as defined in (5.16). For the design example 1, Figure 5.11 shows the error signal obtained after the time domain simulation. We can see that the error is bounded by ±0.025. Moreover, we also compute the error rms which is \( \text{rms}[e(k)] = 0.0093 \). The design approach proposed is validated since we obtain similar results to the ones obtained in [44] (\( H_\infty \) norm of 0.0124 and filters with 11 modes). These results are to be expected since the analysis filter are power complementary and also a explicit solution can be found not [2] for perfect reconstruction of the input signal.

5.2.2 Synthesis Filters Design Example 2

For the design example 2 we select the analysis filter \( H_0(z) \) to be a low pass Finite Impulse Response (FIR) linear phase filter of order 19 and cutoff frequency of \( \pi/2 \). Such a filter is obtained by using the Matlab\textsuperscript{®} command \texttt{fir1} [64]. Its transfer
function is as follows:

\[ H_0(z) = 0.0019 + 0.0028z^{-1} - 0.0053z^{-2} - 0.0100z^{-3} + 0.0174z^{-4} + 0.0288z^{-5} \]
\[ -0.0465z^{-6} - 0.0764z^{-7} + 0.1413z^{-8} + 0.4459z^{-9} + 0.4459z^{-10} \]
\[ +0.1413z^{-11} - 0.0764z^{-12} - 0.0465z^{-13} + 0.0288z^{-14} + 0.0174z^{-15} \]
\[ -0.0100z^{-16} - 0.0053z^{-17} + 0.0028z^{-18} + 0.0019z^{-19} \]  \hspace{1cm} (5.17)

We also define, as in example 1, the high pass analysis filter to be \( H_1(z) = H_0(-z) \). Figure 5.12 shows the magnitude bode plot of the analysis filters versus the normalized frequency. Again, we can see the low and high pass behavior which is given by the definition of both filters.

The synthesis filter bank is designed by following the approach derived in Section 5.1.1 using \( m = 21 \) and obtaining an optimal \( H_\infty \) norm of 0.0229. The filters obtained are depicted in Figure 5.13.

Following the same steps as in the first example, we evaluate the performance of the two channel filter bank by a time domain simulation. The signal \( x(k) \) defined
Figure 5.12: $|H_0(z)|$ and $|H_1(z)|$ filters versus normalized frequency for the design example 2.

Figure 5.13: $|F_0(z)|$ and $|F_1(z)|$ filters versus normalized frequency for the design example 2.

in (5.16) is used as a input signal of the filter bank and the error signal $e(k)$ is measured. Figure 5.14 plots the error signal $e(k)$ obtained in the simulation. We can see that it is bounded by ±0.08. Moreover, we also compute the error rms which
is $rms[e(k)] = 0.0212$.

![Graph showing $e(k)$ error signal after simulation for design example 2.](image)

Figure 5.14: $e(k)$ error signal after simulation for the design example 2.

Finally, we also check the complexity of the synthesis filters designed. The synthesis filter $F_0(z)$ has 22 states while the $F_1(z)$ synthesis filter has 23. In this example, the analysis filters are not power complementary thus the perfect reconstruction solution in [2] does not apply. However, the synthesis filter is designed with the proposed design method achieving a small $\|M(z)\|_\infty$. Furthermore, the comparison of the results with the ones in [44] ($H_\infty$ norm of 0.0296 and both filters with 23 modes) also validate the design approach presented. With smaller filter complexity the signal error between the reconstructed and the desired signal is reduced from a 2.9% to a 2.2%.

### 5.2.3 Design Example 3: Iterative Design

The third example is motivated for a general case where the initial analysis filters are randomly selected. In this case, we use two fifth order analysis filters $H_0^{(0)}(z)$ and $H_1^{(0)}(z)$ to start the filter bank design. Figure 5.15 plots their magnitude bode
plot. Notice that now $H_1^{(0)}(z) \neq H_0^{(0)}(-z)$ and that $H_0^{(0)}(z)$ has higher gain in higher frequencies while it is the opposite for $H_1^{(0)}(z)$. The imposed condition ($H_1^{(0)}(z) \neq H_0^{(0)}(-z)$) forced the analysis filters to be band pass and cover the whole frequency spectrum.

Figure 5.15: $|H_0(z)|$ and $|H_1(z)|$ filters versus normalized frequency for the design example 3.

Using the above mentioned analysis filters $H_0^{(0)}(z)$ and $H_1^{(0)}(z)$ we design the filter banks using the iterative design approach presented in Section 5.1.3 for a delay $m = 4$. The optimization process needs five iterations (i.e., eleven designs) to obtain the filter bank, ending with a $\mathcal{H}_\infty$ error norm of $5.9763 \cdot 10^{-6}$. Table 5.1 shows the evolution of the $\|M(z)\|_\infty$ at each iteration. Furthermore, the filters designed $H_0(z)$, $H_1(z)$, and $F_0(z)$ have ten modes while the filter $F_1(z)$ has eight modes.

Figure 5.15 and Figure 5.16 plot the magnitude bode plot of the designed analysis and synthesis filters respectively. We can see in Figure 5.15 how the analysis filters have changed their gains to a more similar low/high pass behavior. Notice also, that their plots do not cross in the normalized frequency 0.25 as happened in the first two design examples. On the other hand, Figure 5.16 shows that the synthesis filters
Table 5.1: $\|M(z)\|_\infty$ for the each iteration in the design example 3

<table>
<thead>
<tr>
<th>Iteration</th>
<th>$|M(z)|_\infty$</th>
<th>Iteration</th>
<th>$|M(z)|_\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2384</td>
<td>3b</td>
<td>2.2606 $\cdot 10^{-4}$</td>
</tr>
<tr>
<td>1a</td>
<td>0.0246</td>
<td>4a</td>
<td>9.1130 $\cdot 10^{-5}$</td>
</tr>
<tr>
<td>1b</td>
<td>0.0109</td>
<td>4b</td>
<td>3.6672 $\cdot 10^{-5}$</td>
</tr>
<tr>
<td>2a</td>
<td>0.0041</td>
<td>5a</td>
<td>1.4851 $\cdot 10^{-5}$</td>
</tr>
<tr>
<td>2b</td>
<td>0.0023</td>
<td>5b</td>
<td>5.9763 $\cdot 10^{-6}$</td>
</tr>
<tr>
<td>3a</td>
<td>$6.0376 \cdot 10^{-4}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

have a shape that does not emulate a low/high pass filtering behavior. However, these filters achieve an error norm of the $10^{-6}$ order for a small delay (i.e., $m = 4$).

Finally, as in the other two design examples we also evaluate the performance of the filter bank with a time domain simulation. Figure 5.17 shows the error signal $e(k)$. We can see that the error is very small ($\simeq 10^{-5}$). In addition, we have that $rms[e(k)] = 4.1635 \cdot 10^{-6}$. 
5.3 Conclusions

This Chapter introduces an alternative approach for the design of multirate filter banks. Such approach is based on the lifting technique which is used to represent multirate systems with LTI systems. The lifted systems are used to formulate a model matching problem that leads to a design of either the analysis or synthesis filter banks. Such problem uses as a reference model a $m$ delay ($z^{-m}$). Moreover, the problem is reformulated to accommodate the constraint of the filters being stable. The transformed problem is a General Distance Problem, which has causality constraints due to the lifting approach used and it is solved as presented in Chapter 4 of this dissertation.

The design approach proposed can be used to design synthesis (analysis) filters based on the prior knowledge of the analysis (synthesis) filter bank. The combination of both methods leads to an iterative process that can be used to design both filters for the same problem. Three examples are shown to validate the proposed design
methodology. In the first two examples (same as in [44]), the analysis filters are given while in the third example the analysis filter is selected randomly and the iterative design process is used to design the multirate filter bank.

The performance of the multirate filter bank is evaluated by the $\mathcal{H}_\infty$ norm of the error signal $e(k)$, which is defined as the difference between the desired signal $x(k - m)$ and the reconstructed signal $\hat{x}(k)$. The results show the norm to be very small thus it can be said that perfect reconstruction is achieved in the multirate filter bank. The filters proposed here, have equal or better performance to those designed in [44]. Additionally, their complexity is substantially smaller. Finally, a time domain simulation is also performed thus computing the error signal $e(k)$ for a sinusoidal input signal $x(k)$ (5.16). The signal obtained in the simulation and its root mean square value show that very little error is created by the designed multirate filter banks.
Chapter 6

Summary and Conclusions

In this Dissertation we have reviewed the fundamental operators used in multirate digital signal processing such as upsamplers, downsamplers, delays, and advances. Furthermore, we have arranged the basic operators to obtain more complex ones, as the lifting and inverse operators. The lifting technique in time domain concatenates a signal after rearranging its content by increasing its dimension. In addition, we have used such operators to obtain different representations of multirate sampled data systems.

Using the lifting approach, we can compute an equivalent LTI representation of multirate systems and the polyphase approach leads to a description of the multirate system that is based on the polyphase components of the corresponding fast rate system. Moreover, we also have derived the computation of the frequency response matrix of multirate systems. Such matrix is computed based on the $N$-periodicity and LPTV characteristics of a multirate system. It uses the same approach as in the lifting technique but it is derived in the frequency domain. The frequency response matrix also helps us to evaluate the amount of energy transfer between frequency bands that is associated to a multirate system, also known as aliasing. More precisely, it can be used to compute the distance (in the $\mathcal{H}_\infty$ norm sense)
between a multirate system and a fast rate systems. In addition, by computing the $\mathcal{H}_\infty$ norm of some of its elements, we can evaluate the bounds of the distance of a multirate system to the subspace of LTI systems.

The representations obtained are used in the design of multirate $\mathcal{H}_\infty$ optimal controllers and filters. Moreover, the frequency response matrix and its properties are used to analyze and evaluate the performance of the designed $\mathcal{H}_\infty$ optimal controllers.

A two-stage design methodology is presented for the design of $\mathcal{H}_\infty$ optimal multirate controllers. These controllers become multirate because the low frequency modes are implemented at a slower rate. A key feature of the proposed approach is that the redesign of the slow rate part of the controller rather than simply using resampling.

The slow rate design is effected on the plant modified by the high frequency part of the controller operating at the fast sampling rate and by using lifting. The high and low frequency decomposition of the fast rate controller is obtained for either a series or a parallel configuration. Such decomposition is derived for SISO and MIMO systems. The proposed approach addresses potential problems of aliasing due to decimation of the slow rate part of the controller by taking into consideration closed-loop performance rather than merely approximating the controller in an open-loop sense. The two-stage approach can be thought as an automated approach to produce the customary notch filters used to reduce aliasing in a systematic and optimal manner within the $\mathcal{H}_\infty$ problem formulation.

The two-stage design approach and the resampling approach, in which the low frequency part of the controller is downsampled before implementation, are compared on two design examples. Our approach achieves better results both in terms of minimizing aliasing in the closed-loop system measured by computing the distance of the closed-loop multirate system design to the fast rate closed-loop system
and in terms of closed-loop performance measured by comparing the time-domain closed-loop responses of the multirate designs to the performance of the fast rate system.

The results obtained show that the proposed design methodology leads to multirate controllers achieving performance close to that of fast rate controllers at an overall computational load similar to multirate controllers designed by resampling the low frequency modes of the controller. It has been also shown that the multirate controller designed using our approach exhibit similar performance in time domain. In the systems presented, the two stage multirate controller using series configuration obtains a similar response when an impulse disturbance is applied in the measurement to the one computed when using the fast rate controller.

On the other hand, in the frequency domain we have used the frequency response matrix to help us analyze and measure some characteristics of the multirate systems that we have obtained. We can compute the distance between the fast rate closed loop and the multirate closed loop. Since the designed multirate controller is able to compensate any possible aliasing due to downsampling, its effect is minimized and we are close to the fast rate closed loop system. The results also show that both series and parallel configuration multirate controllers are closer than parallel or series configuration controller implemented using the resampling approach.

Moreover, we also study the computational load reduction and the closed-loop performance measure (i.e. distance to the fast rate closed-loop) as we increase the multirate ratio $N$ (i.e., slow sampling time over fast sampling time). The results show a reasonable trade off between performance and computational load for small $N$ when using the multirate $H_{\infty}$ optimal controller in series configuration.

We have also proposed a design algorithm for the design of $H_{\infty}$ optimal causal multirate controllers. The design methodology is based on the parametrization of all stabilizing controllers, also known as the Youla parametrization. We review the
parametrization for an $\mathcal{H}_\infty$ optimization problem and reformulate it as a General Distance Problem. Furthermore, we also analyze the causality constraint in multi-rate systems. We translate such constraint to the feedthrough term of the Youla parameter obtained in the controller parametrization and reformulate the $\mathcal{H}_\infty$ optimal control problem as a constrained optimization problem. Such optimization problem is solved using the Ellipsoid algorithm, which is also reviewed. The Ellipsoid algorithm computes a sequence of ellipsoids of decreasing volume that converge to a local optimal solution if the initial ellipsoid used contains a solution of the problem. We validate the proposed design approach with an example. The results show that using a causal $\mathcal{H}_\infty$ optimal multirate controller we can obtain similar performance to that of a fast rate $\mathcal{H}_\infty$ optimal controller with the advantage of a reduction of its computational load.

Finally, we study the design of filter banks via $\mathcal{H}_\infty$ optimization. The problem is formulated as a model matching problem using as a reference model a $m$ delay $z^{-m}$. Current approaches are based on the used of polyphase matrices. Such approach uses signal content from the previous time instances, which has the advantage of not requiring to constrain the filters to be causal but has the disadvantage of using less updated input information for obtaining its output. That could lead to worst performance in some applications. The design approach presented is based on the lifting technique and the reformulation of the problem as a general distance problem. The problem is solved by constraining the lifter filter bank to be causal. We propose a design approach for the design of either the analysis or the synthesis filter bank. Moreover, an iterative procedure is also proposed for the design of both filter banks. The proposed approach can be seen as a multirate $\mathcal{H}_\infty$ filtering approach. The validation of the method is done through three design examples. The results obtained in the first two examples show that the proposed methodology obtains equal or smaller $\mathcal{H}_\infty$ optimal norm in the design process and also equal or less com-
putational complexity in the synthesis filters designed. The third example evaluates the iterative design algorithm proposed. The results show that the $\mathcal{H}_\infty$ norm of the error relating the input signal and the difference between the desired and the reconstructed signal is very small. In addition, time domain simulations show very small error signal for a sinusoid input signal. These results validate the proposed approach for the design of filter-banks.

**Proposed Future Research Directions**

Although in this Dissertation we propose algorithms for the design of causal multirate $\mathcal{H}_\infty$ optimal controllers and filters, we also think that further research can find new applications in which they can be used. Also developing new connections between the control theory and the digital signal processing theory can open new research paths that have not been followed yet. Moreover, further research in the area can lead to experimental studies in practical applications pursuing a more detailed study of the computational load and implementation characteristics of causal $\mathcal{H}_\infty$ multirate controllers or filters. The proposed filter bank design approach also opens a possibility of continuing the research in the multirate $\mathcal{H}_\infty$ filtering. In addition the study and expansion of the proposed methodology and design algorithms for nonuniform filter banks seems to lead to a promising research direction.
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